SPRING 26 - CALCULUS 3 - EXAM 3 - Solutions

1) Find the volume of the figure that is above the xy-plane, inside the cylinder given by $x^2 + y^2 = 4$, and below the paraboloid given by $z = x^2 + y^2$.

We choose cylindrical coordinates for this problem. The floor of the figure is the circular cross section (radius 2) of the cylinder in the xy-plane The paraboloid and the cylinder intersect in a circle of radius 2 in the plane at $z = 4$. By the rotational symmetry of the figure about the z-axis, the limits of integration are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. The paraboloid is the "roof" of the figure, so $0 \leq z \leq x^2 + y^2 = r^2$. We have the volume integral:
$\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{r^2} rdzdrd\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}[z]_0^{r^2} rdrd\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3 drd\theta$. *This becomes*
$\int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = 4 \cdot 2\pi = 8\pi$.

2) A thin circular disk is centered at the origin and extends from $x = -2$ to $x = 2$. The (area) density of the disk is given by $\sigma(x,y) = e^{-a(x^2+y^2)}$ in grams per sq cm. What is the total mass of the disk in grams?

Switching to polar coordinates, we have a circle of radius 2 centered at the origin with density function $\sigma(r) = e^{-ar^2}$. Total mass $M = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\sigma(r)rdrd\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} e^{-ar^2} rdrd\theta$. For the r integral $\int_{r=0}^{2} e^{-ar^2} rdr$, let $u = -ar^2$, then $du = -2ardr$, so the integral becomes
$\int_{r=0}^{2} e^u\left(\frac{du}{-2a}\right) = \frac{-1}{2a}\int_{r=0}^{2} e^u du = \frac{-1}{2a}\left[e^{-ar^2}\right]_{r=0}^{2}$. *This evaluates to $\frac{-1}{2a}[e^{-4a} - 1]$. Since $a > 0$ for there to be any physical mass, we can rewrite this as $\frac{1 - e^{-4a}}{2a}$. Finally,*
$M = \frac{1 - e^{-4a}}{2a}\int_{\theta=0}^{2\pi} d\theta = \frac{\pi(1 - e^{-4a})}{a}$.

3) Find the centroid of the figure bounded by the x-axis and the curve $y = \sqrt{a^2 - x^2}$.

This is the part of the disk of radius a centered at the origin in the upper half plane. Since it is symmetric about the y-axis, the coordinate $\bar{x} = 0$. The only remaining question is the value for $\bar{y}$. This is given by finding the moment of the half-disk area about the x-axis:
$M_x = \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} ydydx$. *We have* $M_x = \int_{x=-a}^{a}\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_{x=-a}^{a}(a^2 - x^2)dx$. *This reduces to* $M_x = \frac{1}{2}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{2}\left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right] = \frac{1}{2}\left(\frac{4a^3}{3}\right) = \frac{2a^3}{3}$. *The area A of the half-disk is $\frac{\pi a^2}{2}$, so* $\bar{y} = \frac{M_x}{A} = \frac{2a^3}{3} \cdot \frac{2}{\pi a^2} = \frac{4a}{3\pi}$. *Then the centroid is at* $\left(0, \frac{4a}{3\pi}\right)$.

4) A quartz crystal occupies the space in the first octant where $0 \leq x \leq 1$, $0 \leq z \leq 1 - x$, and

$0 \le y \le 3 - x - z$. The internal temperature of the crystal is given by the function $T(x,y,z) = e^{x+y}$ in degrees celsius. What is the weighted average of the temperature over the entire crystal?

The volume integral is $\int_0^1 \int_0^{1-x} \int_0^{3-x-z} dydzdx = \int_0^1 \int_0^{1-x} (3 - x - z)dzdx = \int_0^1 \left[3z - xz - \frac{z^2}{2} \right]_0^{1-x} dx$. *This reduces to* $\int_0^1 \left(3(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right) dx = \frac{7}{6}$. *Now we need the temperature weighted integral:* $\int_0^1 \int_0^{1-x} \int_0^{3-x-z} T(x,y,z)dydzdx = \int_0^1 \int_0^{1-x} \int_0^{3-x-z} e^{x+y}dydzdx$. *This is similar to the volume integral and upon integrating with respect to* y *becomes* $\int_0^1 \int_0^{1-x} (e^{3-z} - e^x)dzdx$. *Doing the* z integral gives $\int_0^1 (e^3 + xe^x - e^{2+x} - e^x)dx$. Finally the x integral gives $2 + e^2 - e$. *Finally, the average temperature in the crystal is* $\overline{T} = \dfrac{2 + e^2 - e}{\frac{7}{6}} \approx 5.717$ *deg C.*

5) Find the moment of inertia of a solid sphere of radius R and mass density σ grams per cubic cm about a diameter.

Spherical coordinates are appropriate here. Suppose we place the diameter on the z axis from $z = -a$ to $z = a$. The distance of a differential mass dm in the sphere at the point (ρ, θ, ϕ) to the z axis would be $\rho \sin\phi$ for any θ. So the differential contribution to moment of inertia about the axis would be $dI = (\rho \sin\phi)^2 dm$. The differential mass $dm = \sigma\rho^2 \sin\phi d\rho d\theta d\phi$. So $dI = \sigma\rho^4 \sin^3\phi d\rho d\theta d\phi$. Then $I = \sigma \int_0^\pi \int_0^{2\pi} \int_0^R \rho^4 \sin^3\phi d\rho d\theta d\phi$. Doing the ρ integral we get $I = \sigma \int_0^\pi \int_0^{2\pi} \left(\frac{R^5}{5}\right) \sin^3\phi d\theta d\phi = \frac{\sigma R^5}{5} \int_0^\pi \int_0^{2\pi} \sin^3\phi d\theta d\phi$. Doing the θ integral reduces this to $I = \frac{\sigma R^5}{5} \int_0^\pi (2\pi) \sin^3\phi d\theta d\phi = \frac{2\sigma\pi R^5}{5} \int_0^\pi \sin^3\phi d\phi$. Finally, since $\int_0^\pi \sin^3\phi d\phi = \frac{4}{3}$, we have $I = \frac{4}{3} \cdot \frac{2\sigma\pi R^5}{5} = \frac{8\sigma\pi R^5}{15}$. Since the overall mass M of the sphere is $\frac{4\sigma\pi R^3}{3}$, you can write $I = \frac{4\sigma\pi R^3}{3} \cdot \frac{2R^2}{5} = \frac{2}{5} MR^2$.