Logic - Lecture 22

In 1931 Kurt Gödel, an Austrian logician, introduced several theorems relating to "incompleteness." The first of these states that a logic (or logistic) system which is strong enough to include the integers is flawed in the following way: no family of (consistent) axioms within the system, where the theorems based on those axioms can be listed by some algorithmic procedure, is capable of proving all true statements about the arithmetic of $\mathbb{N}$.

We should establish some definitions:

1. A logic or deductive system is <u>rigorous</u> or has <u>rigor</u> when only formulas that are logically entailed by its axioms are allowed to be theorems. Ultimately this means the symbols (like $p \supset q$) are uninterpreted and the rules to use them, the principles of inference, are explicit (and usually few). Systems that are not rigorous have unstated assumptions in them. So a logistic system is a formal deductive system which has rigor.

② A logistic system deals with <u>WFF's</u> : <u>well-formed formulas.</u> We know that $p \supset (q \supset r)$ is a valid formula in our predicate calculus, but $pq \supset \supset r$ is not. A non-formal deductive system like Euclidean geometry is an arrangement of propositions about space where the subject may be discussed. By understanding the language, we can sort out statements as being meaningful or nonsensical. We desire our deductive system to be based on abstract WFF's that make sense when we endow them with specific meaning.

So for us a logistic system will have five properties:

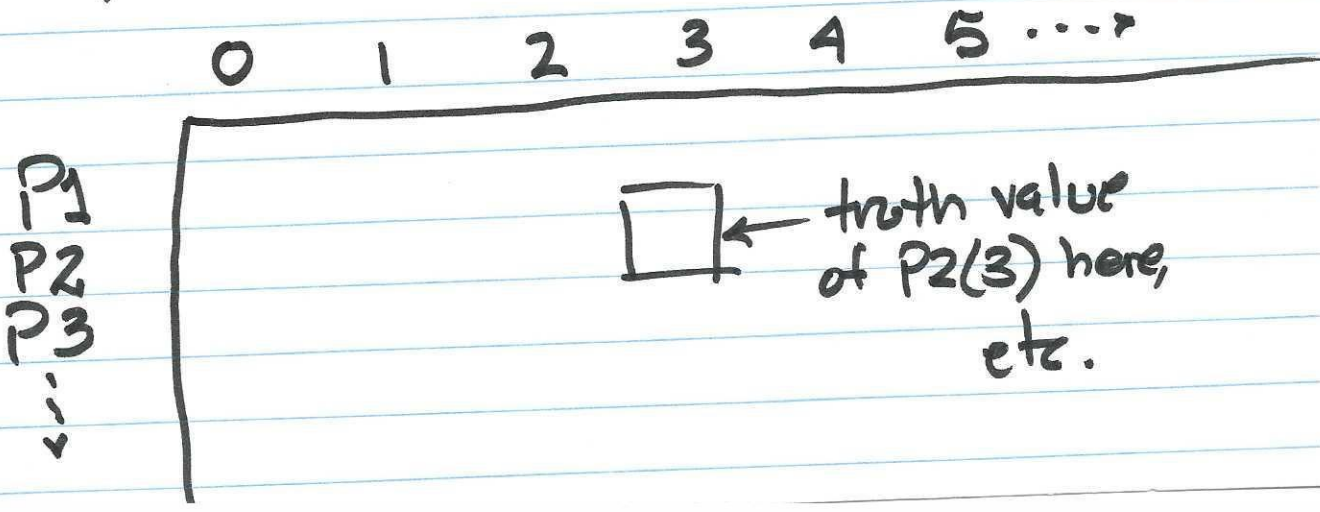
1. a list of primitive (undefined) symbols which together with any symbols defined in terms of them will constitute the only symbols allowed in the system;
2. a purely formal (syntatical and not semantic, i.e. structure and not meaning) criterion for determining sequences of allowed symbols as WFF's or non-WFF's;
3. a list of WFF's considered to be axioms;
4. a purely formal criterion for deeming sequences of WFF's as either <u>valid</u> or <u>invalid</u> arguments; and

5 a purely formal criterion based on 3 and 4 for distinguishing theorems from non-theorems of the system.

The following is a heuristic attempt at demonstrating Gödel's incompleteness theorem.

Again, incompleteness means there are true theorems expressible in the logistic system which cannot be proved in the system.

Let us construct the following semi-infinite array A:



We have the non-negative integers across the top and a list of propositions about those integers down the side. If proposition (theorem) P_n is true for integer m, we write T in the (n,m) entry for A. Otherwise, F. A typical proposition might be "m is a prime" or "m is a square." Now we need to "algorithmically list" the propositions we can make about $\mathbb{N}_{0+}$. There are 256 ASCII characters available to express propositions. Let us also number the primes in order of increasing magnitude: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.

We are going to use the primes to code for position in a statement and we will use the ASCII symbol number as the power of the prime which defines its position.

For example, take "m_is_prime" as a theorem. There are 10 characters, so we write $2^{A(m)} 3^{A(\_)} 5^{A(i)} 7^{A(s)} 11^{A(\_)} \cdots$

$13^{A(p)} 17^{A(r)} 19^{A(i)} 23^{A(m)} 29^{A(e)} = N.$

N is going to be big, obviously, but it also gives us an injective function from theorems to $\mathbb{N}$. Do you see why?

Fundamental Theorem of Arithmetic!

Here A(m), say is the ASCII symbol number for m.

Now encode all theorems about N_{0+}.

These theorems are all finite strings.

There may be infinitely many, but at least they are denumerable. Let us arrange them by increasing size of N, the prime/ASCII derived expression.

Every one of these theorems is either true or false for a particular m.

Consider the big array A again. In the (n,n) position of that array we have a T if proposition (theorem) P_n is true for n (i.e. $P_n(n) = T$) or F if $P_n(n) = F$.

Here is a new theorem: $P_x(n) = F$ if $P_n(n) = T$ and $P_x(n) = T$ if $P_n(n) = F$.

There is no question that P_x is well-defined and we can determine $P_x(n)$ from its definition, so the truth of P_x as a theorem is established. But this is external to the system of theorems derivable from the original axioms of our system - <u>because P_x is not a line item of the array</u>. If it were, say at the k^{th} line, then the (k,k) entry of A would be both true and false. The contradiction allows us to conclude that the truth of P_x for all $\mathbb{N}_{0+}$ is decidable, but not from any argument based on the axiom set.

Of course you could add Px as a new axiom, but then the argument can be constructed again. So every axiom set cannot reach every true statement.

For a while mathematicians thought Fermat's Last Theorem might be a proposition like P_x, but then of course Wiles et al proved it in 1995. There is always the Collatz Conjecture.