MG8X5

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Limits of multivariate functions:

We say $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0) and we denote that as "naught": 0

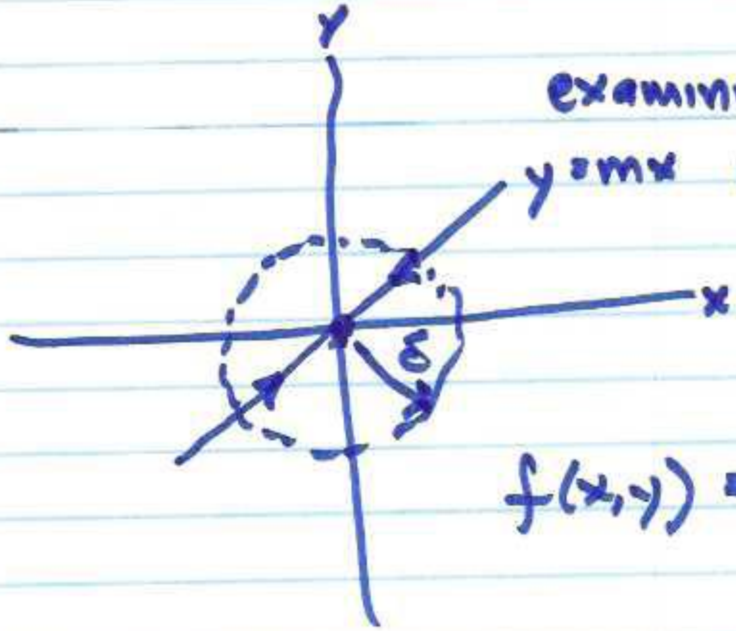
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if given $\varepsilon > 0$, there exists a corresponding number $\delta(\varepsilon) > 0$ such that for all ($\forall$) $(x,y) \in \operatorname{dom} f$ the following happens:

$$|f(x,y) - L| < \varepsilon \text{ whenever}$$

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Counterexample : $f(x,y) = \dfrac{x^2+y^2}{xy}$



examine limit of $f(x,y)$
$y=mx$ @ $(x_0,y_0)=(0,0)$

$$f(x,y) = \frac{x^2+m^2x^2}{x \cdot mx}$$

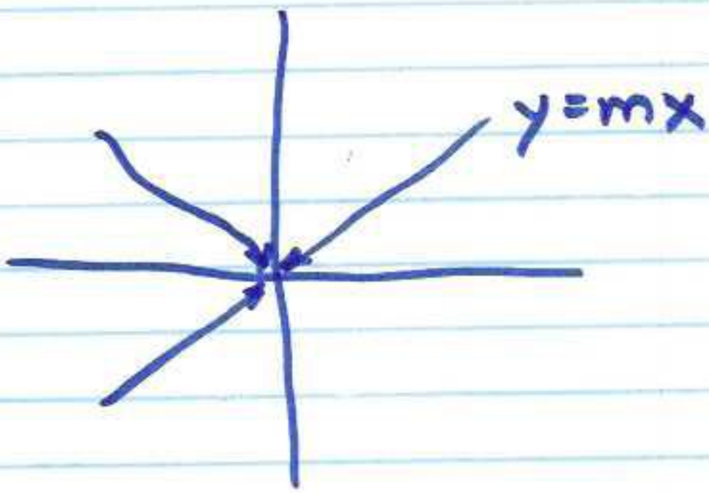
$$= \frac{\cancel{x^2}(1+m^2)}{\cancel{x^2}\,(m)} = \frac{1+m^2}{m}$$

Example where def^n gives a limit

$$f(x,y) = \frac{2xy}{x^2+y^2} \quad (x,y) \neq (0,0)$$

(same phenomenon as above)

Find $\displaystyle \lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = f(x,y)$



$f(x,y)$ along $y = mx$ is:

$$\frac{3x^2 - m^2x^2 + 5}{x^2 + m^2x^2 + 2} = \frac{(3-m^2)x^2 + 5}{(1+m^2)x^2 + 2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 5/2$$

4 Quotient Rule:

$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = L/M$$

provided $M \neq 0$

5 Power Rule

$$\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y)\right]^n = L^n$$

6 Root Rule

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$$

Omit <u>n even</u> if $L<0$

Application: $$\lim_{\substack{\cancel{(x,y)\to(x_0,y_0)}\\(x,y)\to(0,1)}} \frac{x - xy + 3}{x^2y + 5xy - y^3}$$

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$$\lim_{(x,y)\to(0,1)} = \frac{\lim x - \lim xy + 3}{\lim x^2 y + \lim 5xy - \lim y^3}$$

$$= \frac{0 - 0 + 3}{0 + 0 - 1} = \frac{3}{-1} = -3$$

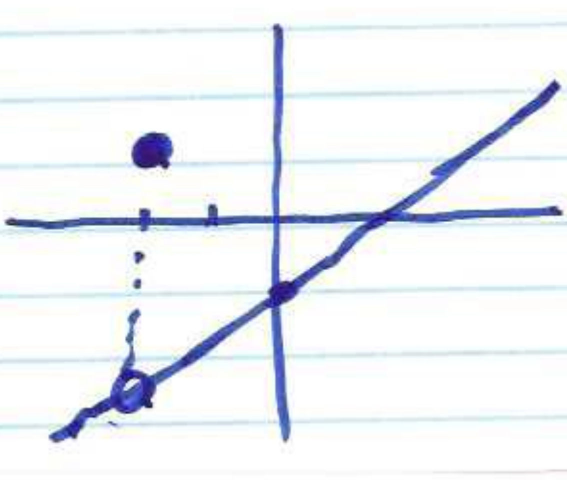
Continuity:

$f(x)$ is <u>continuous</u> @ x_0 if:

(i) $f(x_0)$ exists
(ii) $\lim_{x \to x_0} f(x)$ exists
(iii) (i) = (ii)

$$\frac{1}{x+2}$$

Consider $f(x) = \frac{x^2-4}{x+2} = x-2$



$$f(x) \begin{cases} x-2 & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$$

Rules for continuity:

For all below: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$

1. Sum Rule / Difference Rule

$$\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y) \pm g(x,y)\right) = L \pm M$$

2. Constant Multiple

$$\lim_{(x,y)\to(x_0,y_0)} c\cdot f(x,y) = cL, \text{ any } c\in\mathbb{R}$$

3. Product Rule

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)\cdot g(x,y) = L\cdot M$$

Continuity in 2 (or more) variables

(i) $f(x_0, y_0)$ must exist

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

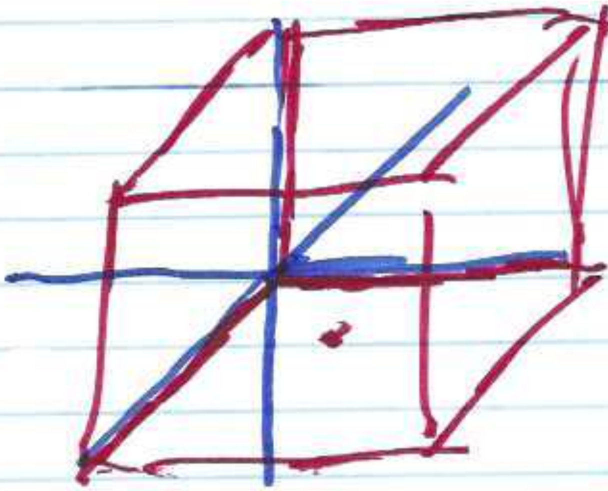
Two-Path Test for Discontinuity

If $f(x,y)$ has different limits approaching (x_0,y_0) then $f(x,y)$ is not continuous @ (x_0,y_0)

Where is $\ln(x+y+z)$ continuous in $\mathbb{R}^3$?

$x+y+z > 0$

$\mathbb{R}^3$ can be decomposed into "octants"



+++ octant

$$f'(x_0) := \lim_{x \to x_0} \left(\frac{f(x_0+h) - f(x_0)}{h} \right)$$

$x - x_0 = h$

$$f_x(x_0, y_0) := \lim_{\substack{x \to x_0 \\ y = y_0}} \left(\frac{\overbrace{f(x_0+h, y_0)}^{f(x)} - f(x_0, y_0)}{\underbrace{h}_{x - x_0}} \right)$$

$x - x_0 = h$

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$$f_y(x_0,y_0) := \lim_{\substack{x \to x_0 \\ y \to y_0}} \left(\frac{f(x_0, y_0+h) - f(x_0,y_0)}{h} \right)$$

$\underline{h = y - y_0}$

$f_x = ?\ \underline{3x^2y^2}$ if $f(x,y) = \underline{x^3y^2}$

$f_y = x^3(2y) = \underline{2x^3y}$

f_x

$f(x,y) = xy \sin x^2 y$

$\underline{= \alpha x \cdot \sin \alpha x^2}$

α constant

$\underline{(\alpha)\sin \alpha x^2} + (\alpha x)$

$\cos \alpha x^2 \cdot 2\alpha x$