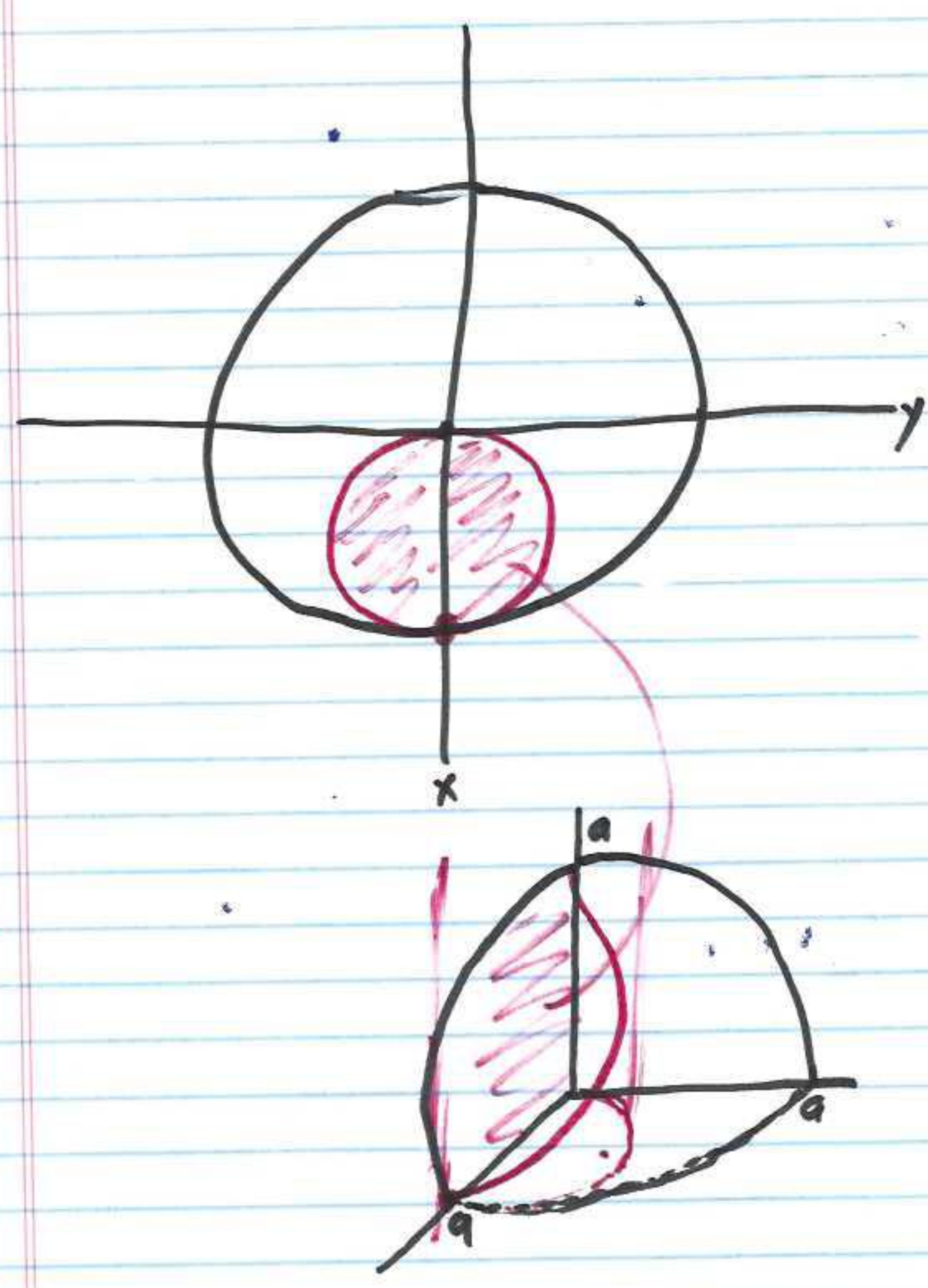
DXWEB



Gauss' Th^{m} (Divergence Th^{m})

Let $\mathbb{F}(x,y,z)$ be a field - think of the velocity field of a moving fluid.

Say $\mathbb{F}(x,y,z) = \underbrace{\rho(\mathbf{r})\mathbf{v}(\mathbf{r})}_{\text{what is dimension}}$

$$\frac{\text{mass}}{\text{length}^3} \cdot \frac{\text{length}}{\text{sec}} = \frac{\text{mass}}{\text{length}^2 \cdot \text{sec}}$$

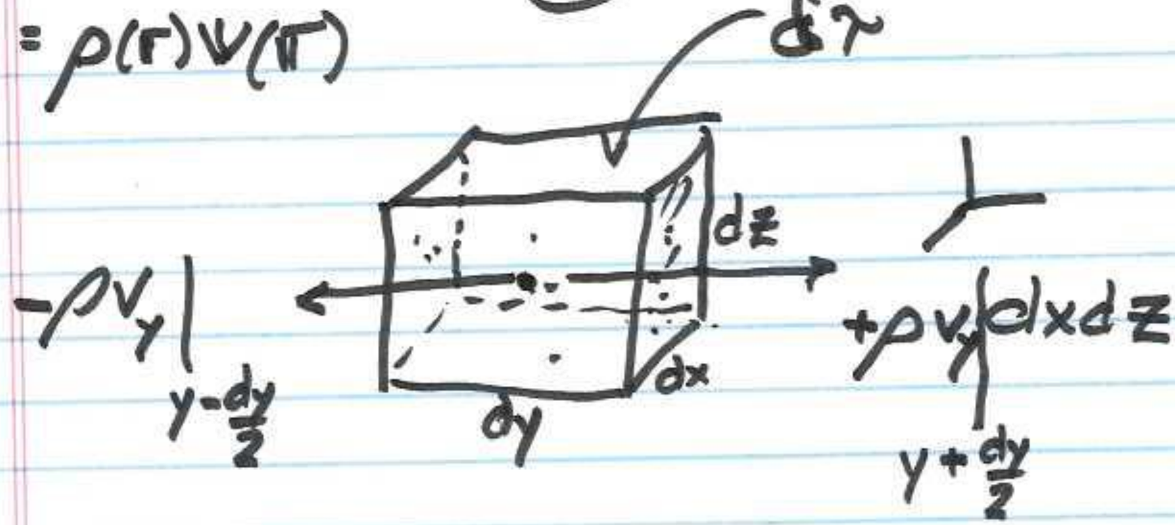
density velocity flux

flux → flow multiply by area

$$\int_V (\nabla \cdot \mathbb{F})\, d\tau = \int_{S_V} \mathbb{F} \cdot \hat{n}\, d\sigma$$

②

$\mathbf{F}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r})$



$= \rho(\mathbf{r})\left[v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right]$

$$\text{net flow} = \underbrace{\left[\rho v_y\Big|_{y+\frac{dy}{2}} - \rho v_y\Big|_{y-\frac{dy}{2}}\right]}_{\frac{\partial}{\partial y}(\rho v_y)\,dy \;=\; d(\rho v_y)} \underline{dx\,dz}$$

So $d(\rho v_y)\cdot dx\,dy\,dz$ = dift flow

$$\iiint_{d\tau} \quad = \iint_{d\sigma} \frac{\partial}{\partial y}(\rho v_y)\,dx\,dz +$$

So net outflow per sec is:

$$\left[\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}\rho v_z\right]dx\,dy\,dz$$

Outflow is $\nabla\cdot(\rho \mathbf{v})\,dx\,dy\,dz = -\frac{\partial \rho}{\partial t}dx\,dy\,dz$

decrease in mat'l in box

So $\nabla\cdot(\rho\mathbf{v})+\frac{\partial\rho}{\partial t}=0$ – no source/sink

Continuity Eqn

Also

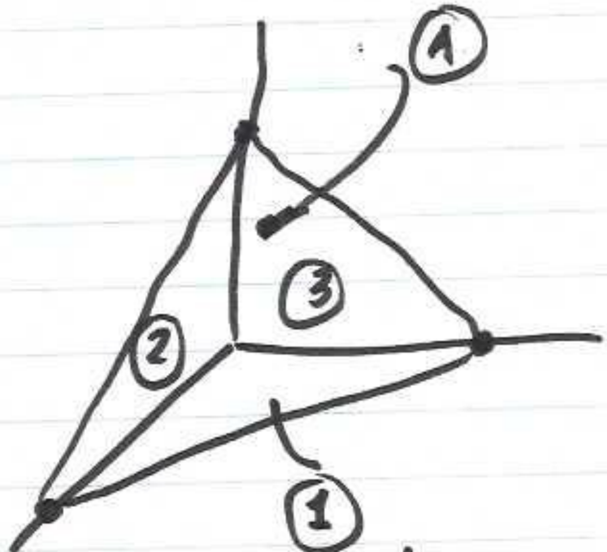
$$\iiint_V \nabla\cdot A\, d\underline{q}^3 = \oiint_{\partial S_V} A\cdot\hat{n}\, d\underline{q}^2$$

sums outflows

sums flow thru boundary

Surface integral:

$$I = \iint_S \mathbb{F} \cdot d\sigma \quad \xleftarrow{} \hat{n}\,dA$$



(1) outward unit normal $\hat{n}$ is $-\hat{e}_z$, so $d\sigma = -\hat{e}_z dA$

$\mathbb{F} = (x+1)\hat{e}_x + y\hat{e}_y - z\hat{e}_z$

Then $\mathbb{F} \cdot d\sigma = 0$ So $I_1 = 0$

(2) On xz plane $y=0$, so $\hat{n} = -\hat{e}_y$

then $d\sigma = -\hat{e}_y dA$ on triangle #2.

$\mathbb{F} = (x+1)\hat{e}_x - z\hat{e}_z$ so $\mathbb{F} \cdot d\sigma = 0$

then $I_2 = 0$

(3) on yz plane $x=0$, so $\hat{n} = -\hat{e}_x$ and

$d\sigma = -\hat{e}_x dA$. On triangle #3,

$$\mathbb{F} = e_x + y\hat{e}_y - z\hat{e}_z$$

Then $\mathbb{F} \cdot d\sigma = \mathbb{F} \cdot -\hat{e}_x dA = -1\, dA$

What is $\iint_{\#3} (-1) dA = (-1)(\frac{1}{2} \cdot 1 \cdot 1) = -\frac{1}{2}$

So $I_3 = 0$

(4) $\hat{n} = (\hat{e}_x + \hat{e}_y + \hat{e}_z) / \sqrt{3}$

Plane is $x + y + z = 1$

direction vector so dir vector is $\langle 1, 1, 1 \rangle$,

then $\hat{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$

then $F\cdot\hat{n}\,dA = \left(\overset{(x+1)}{\hat{e}_x} + y\,\hat{e}_y - z\,\hat{e}_z\right)\cdot\frac{1}{\sqrt{3}}\langle 1,1,1\rangle$

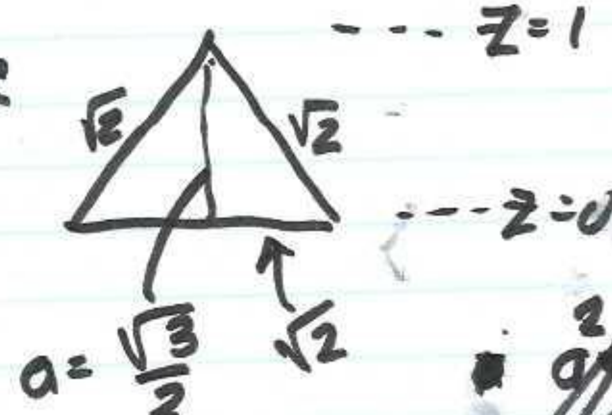
$= \frac{1}{\sqrt{3}}\left(x+1+y-z\right)$ $x+y=1-z$

and $I_4 = \iint_{\Delta_4} \frac{x+1+y-z}{\sqrt{3}}\,dA = \int_{\Delta_4} \frac{2(1-z)}{\sqrt{3}}\,dA$

$I_4 = \int_0^1 2(1-z)^2\,dz = 2/3$

Then $I_1+I_2+I_3+I_4 = 0+0-\frac{1}{2}+\frac{2}{3} = \boxed{\frac{1}{6}}$

Note



$\cancel{a^2+\left(\frac{\sqrt{3}}{2}\right)^2=\sqrt{2}^2}$

$\cancel{a^2+\frac{3}{4}=2} \Rightarrow$

$\mathbb{F} = (x+1)\hat{e}_x + y\hat{e}_y - z\hat{e}_z$ ⑦

altitude

Vol of tetra : $\frac{1}{3}(1)\left(\frac{1}{2}\right) = \frac{1}{6}$

base

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① unit outward normal $-\hat{e}_z$

so $d\sigma = -\hat{e}_z dA$

What is $\mathbb{F} \cdot \hat{n} dA$

$$\left((x+1)\hat{e}_x + y\hat{e}_y\right)\Big|_{z=0} \cdot \left(\hat{e}_z dA\right) = 0$$

② In xz plane

$\hat{n} = -\hat{e}_y$

$$\mathbb{F}\Big|_{A_2} = (x+1)\hat{e}_x - z\hat{e}_z$$

So $\mathbb{F} \cdot \hat{n} dA = 0$

3 On yz plane

$$\hat{n} = -\hat{e}_x$$

$$\mathbb{F} = \hat{e}_x + y\hat{e}_y - z\hat{e}_z$$

$$\mathbb{F}\cdot\hat{n}\,dA = -dA$$

so what is $\displaystyle\iint_{\Delta_3} \mathbb{F}\cdot\hat{n}\,dA = \iint_{\Delta_3} -1\,dA = -\frac{1}{2}$

4 $x+y+z=1$ plane of Δ_4

a normal is $\langle 1,1,1\rangle$ so

$$\hat{n} = \frac{1}{\sqrt{3}}\left(\hat{e}_x + \hat{e}_y + \hat{e}_z\right)$$

$$\mathbb{F}\cdot\hat{n}\,dA = \left[(x+1)\hat{e}_x + y\hat{e}_y - z\hat{e}_z\right]\cdot\left[\frac{1}{\sqrt{3}}\left(\hat{e}_x+\hat{e}_y+\hat{e}_z\right)\right]$$

$$\frac{1}{\sqrt{3}}\left(x+1+y-z\right) =$$

$$2(1-z) = (1-z)+(1-z)$$

$$\nabla \cdot \rho \mathbf{v} = -\frac{\partial \rho}{\partial t}$$

@ point $\underline{\nabla \cdot \rho \vec{v} + \frac{\partial \rho}{\partial t} = 0}$ Continuity Principle

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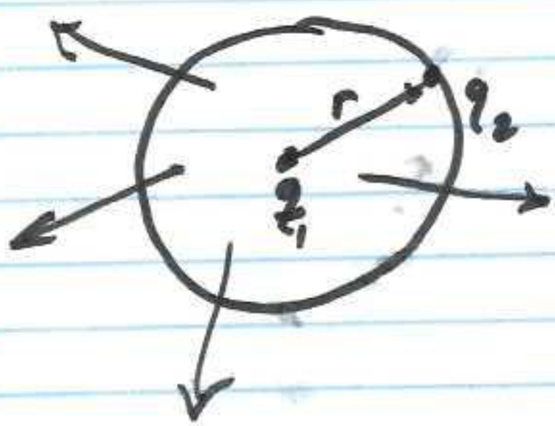
1) $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$

q_1, q_2

$\underline{F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}}$

$$\iiint \nabla \cdot E \, d\tau = \oiint E \cdot \hat{n} \, d\sigma$$



I get

q_1/ε_0

5

$$x^2+y^2+z^2=R^2$$

$2x\hat{i}+2y\hat{j}+2z\hat{k}$ = a normal vector

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{3}}$$

$$\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right) = \hat{n}$$

$$E\cdot\hat{n} = \frac{E_x+E_y+E_z}{\sqrt{3}}$$

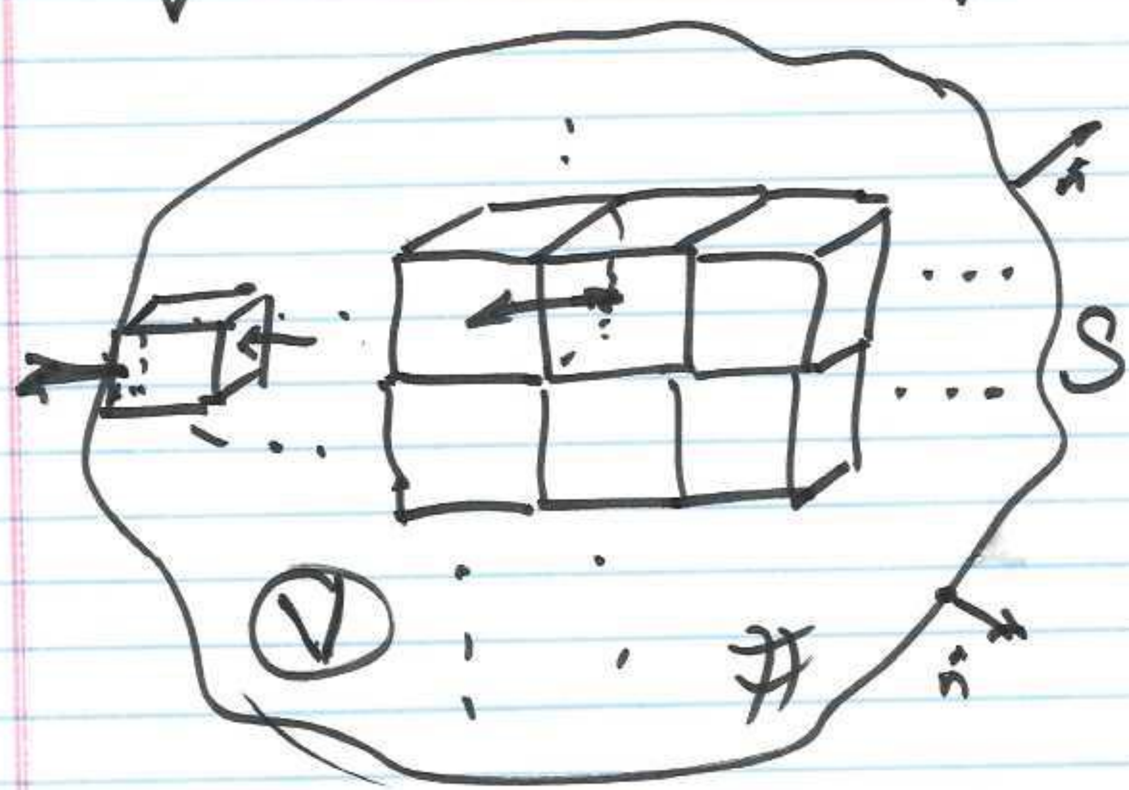
$$E\,4\pi R^2$$

$$\frac{q_1}{\varepsilon_0} = E_{q_1}\cdot 4\pi R^2 \Rightarrow$$

$$E_{q_1} = \frac{q_1}{4\pi\varepsilon_0}\cdot\frac{1}{R^2}$$

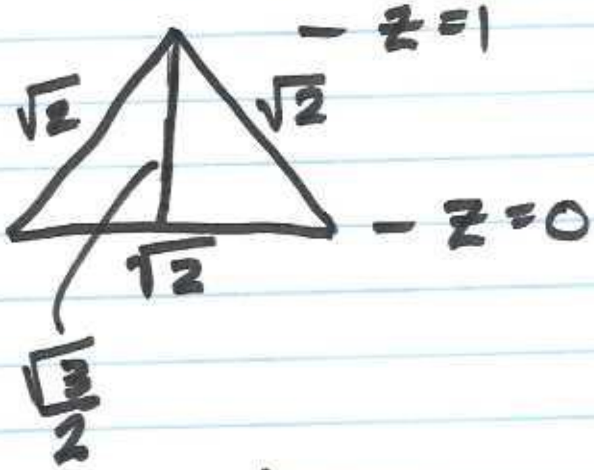
$$\iiint_{d\tau} \rightarrow \iiint \left[\frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z\right] \underbrace{dx\,dy\,dz}_{d\tau}$$

$$\iiint_{\substack{vol \\ V}} (\nabla\cdot\rho(\mathbf{r})\mathbf{v}(\mathbf{r}))\,d\tau = \iint_{\substack{S \\ V}} \rho(\mathbf{r})\mathbf{v}(\mathbf{r})\cdot\mathbf{n}\,dS$$



$$\int_V \nabla\cdot\mathbf{F}\,d\tau = \int_{S_V} \mathbf{F}\cdot\hat{\mathbf{n}}\,d\sigma$$

$$I_4 = \iint_{\Delta_4} \frac{2(1-z)}{\sqrt{3}}\, dA$$



$$dA = \sqrt{2}(1-z)\cdot \tfrac{\sqrt{3}}{2}\, dz$$

$$I_4 = \int_0^1 2(1-z)^2\, dz = \frac{2}{3}$$

so... $I_1 + I_2 + I_3 + I_4$

$$0 + 0 + -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}$$