2SPG3

1/20

Divergence of vector field $\mathbb{F}(x,y,z)$ is a measure of "outwardness"

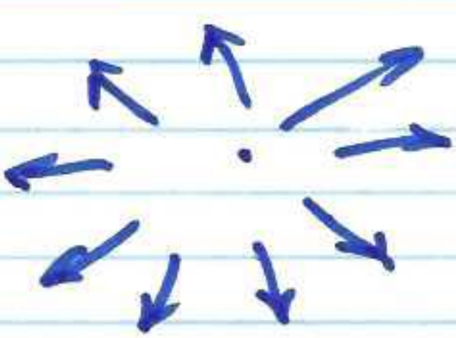
If $\nabla \cdot \mathbb{F} > 0$ source

$\nabla \cdot \mathbb{F} < 0$ sink

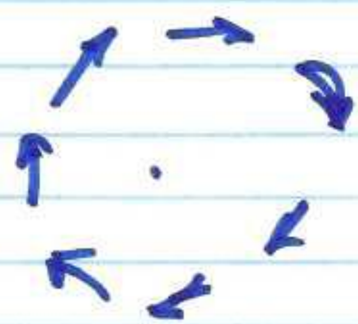
$\nabla \cdot \mathbb{F} = 0$ no net in/out

Curl of vector field $\mathbb{F}(x,y,z)$ is a measure of circulation - tendency to go in closed loops with no source/sink

$\nabla \times \mathbb{F}$ is a vector, unlike $\nabla \cdot \mathbb{F}$, which is a scalar.

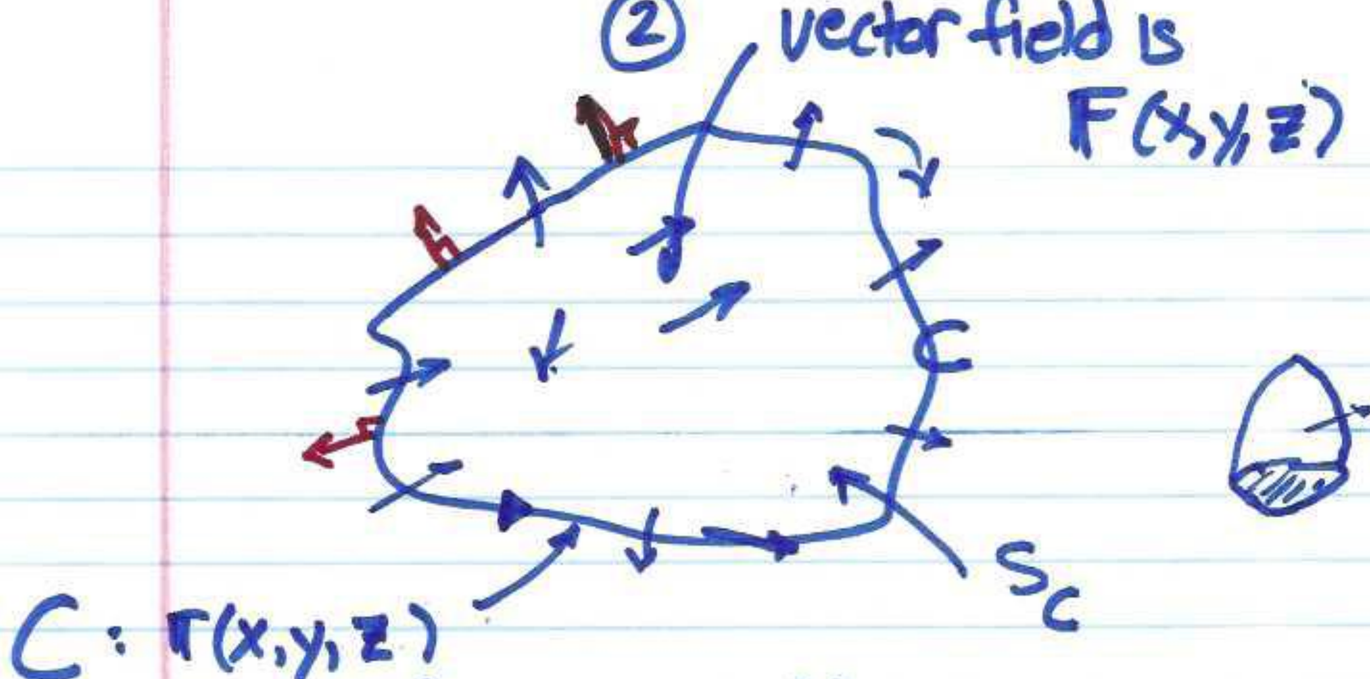


Div $\mathbb{F}$



Curl $\mathbb{F}$

②



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_C} (\nabla \times \mathbf{F}) \cdot \hat{n}\, d\sigma$$

vector differential displacement along C

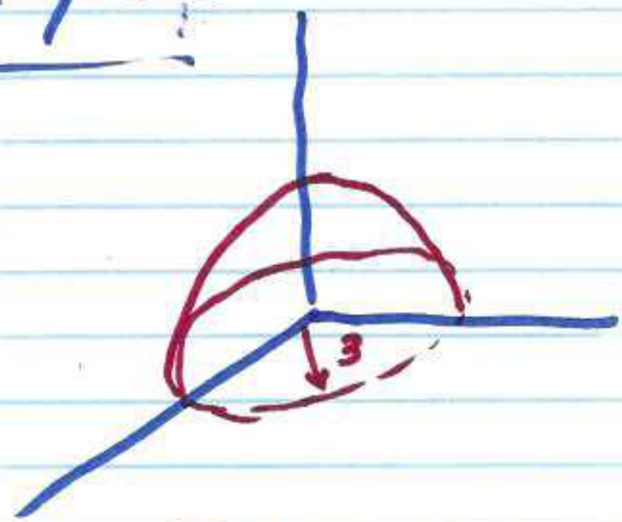
scalar differential area

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Let $S = x^2 + y^2 + z^2 = 9 \quad z \geq 0$

What is bounding circle of S?

$C : \underline{x^2 + y^2 = 0}$



$\underline{\text{Field is}}$ $\underline{\mathbb{F}(x,y,}z) = y\hat{i} - x\hat{j}$

Need to parametrize C:

Let $x = 3\cos\theta$, $y = 3\sin\theta$

$\mathbb{r}(\theta) = 3\cos\theta\,\hat{i} + 3\sin\theta\,\hat{j}$

$d\mathbb{r}(\theta) = -3\sin\theta\,d\theta\,\hat{i} + 3\cos\theta\,d\theta\,\hat{j}$

$\mathbb{F}(\theta) = 3\sin\theta\,\hat{i} - 3\cos\theta\,\hat{j}$

Then $\mathbb{F}\cdot d\mathbf{r} = -9\sin^2\theta\, d\theta - 9\cos^2\theta\, d\theta$

$= -9\, d\theta$

So what is $\oint \mathbb{F}\cdot d\mathbf{r} = \int_0^{2\pi} -9\, d\theta =$

$-9\int_0^{2\pi} d\theta = -18\pi$

Next, find $\nabla \times \mathbb{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} =$$

$\hat{i}(0) - \hat{j}(0) + \hat{k}(-1-1) = -2\hat{k}$

⑤

So $\nabla \times \mathbf{F} = -2\hat{k}$

Now find $\hat{n}$ (outward unit normal)

$$\hat{n} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + 0^2}} =$$

$$\hat{n} = \frac{1}{3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

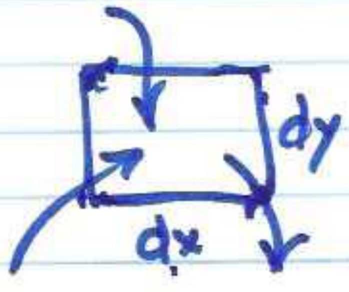
(Ex 7 p. 1029)

From earlier problem $d\sigma = -2\,dA$ ← earlier problem

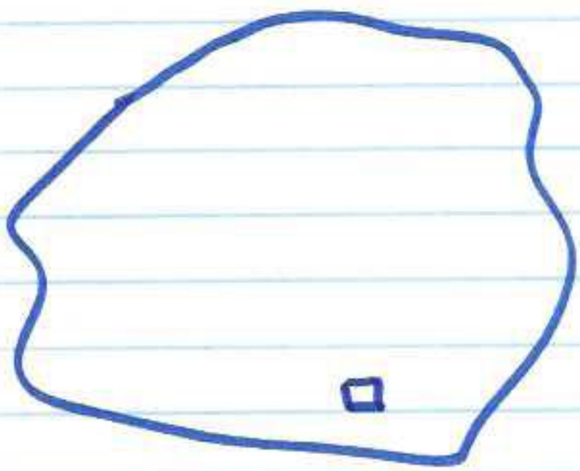
$$\iint (\nabla \times \mathbf{F}) \cdot \hat{n}\, d\sigma = \iint_{\substack{S \\ x^2+y^2=9}} -2\,dA$$

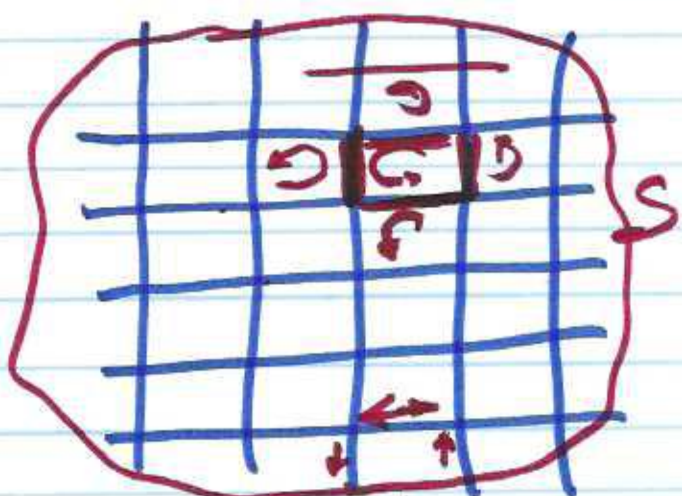
$$-2\int_0^{2\pi}\int_0^3 r\,dr\,d\theta = -2\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^3 = -2\int_0^{2\pi}\frac{9}{2}\,d\theta$$

$$= -9\int_0^{2\pi} d\theta = -9\left[\theta\right]_0^{2\pi} = -18\pi$$



establish for this rectangle





not internally cancelled

Find div $\mathbb{F}$ & curl $\mathbb{F}$ for

$(\nabla \cdot \mathbb{F})$ $(\nabla \times \mathbb{F})$

$\mathbb{F} = ye^{xyz}\,\hat{i} + ze^{xyz}\,\hat{j} + xe^{xyz}\,\hat{k}$

<u>Div</u>:

$$\nabla \cdot \mathbb{F} = y^2 z e^{xyz} + xz^2 e^{xyz} + x^2 y e^{xyz}$$

$$= e^{xyz}\left[zy^2 + xz^2 + yx^2\right]$$

Curl:

$\nabla \times \mathbb{F}$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xyz} & ze^{xyz} & xe^{xyz} \end{vmatrix}$$