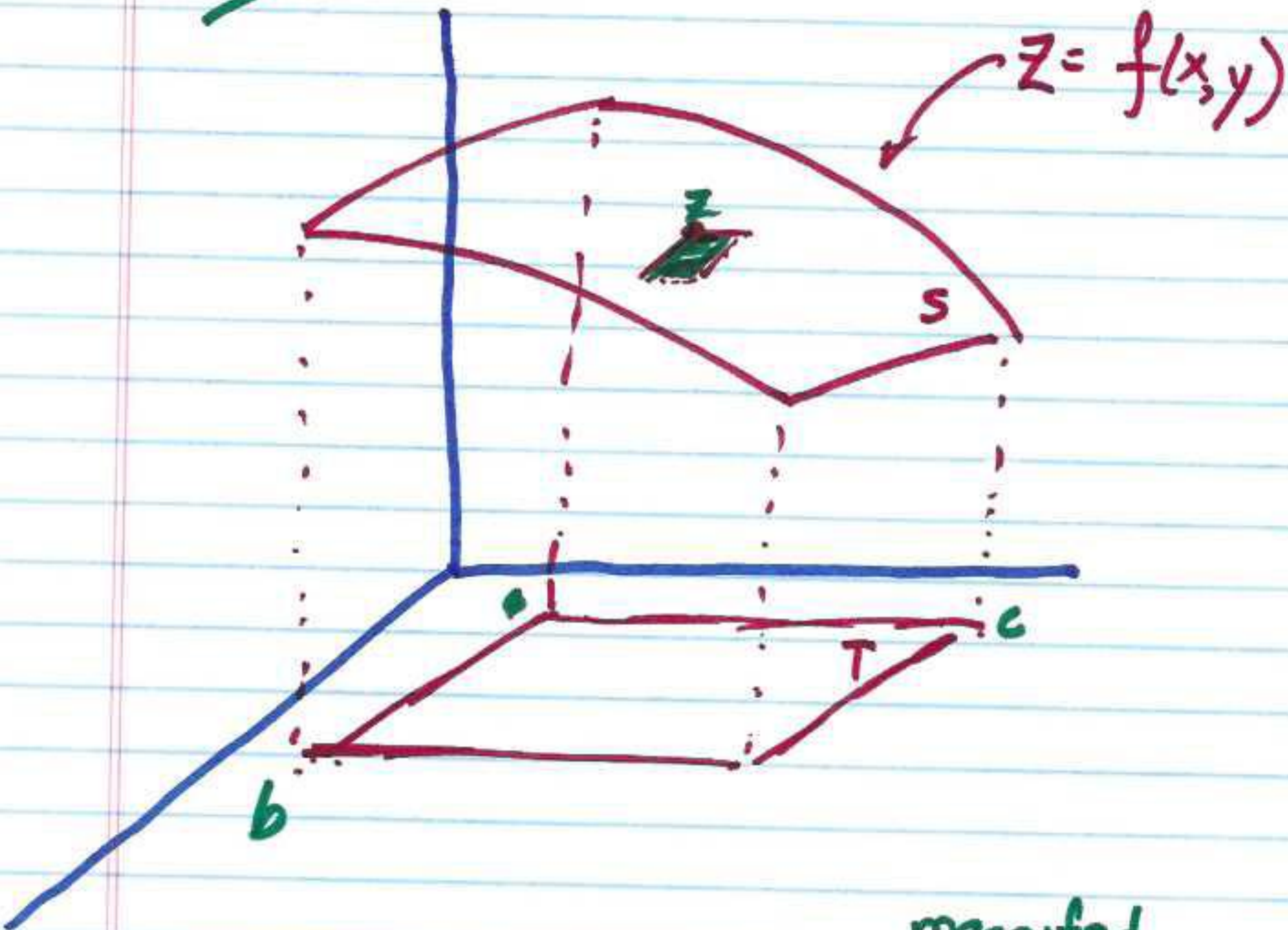
L2NV6

1

4/17

§16.5



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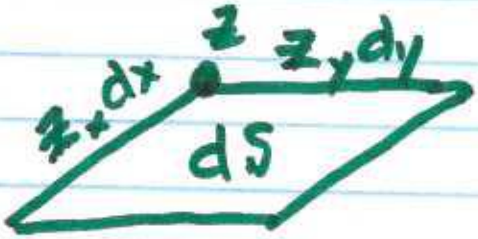


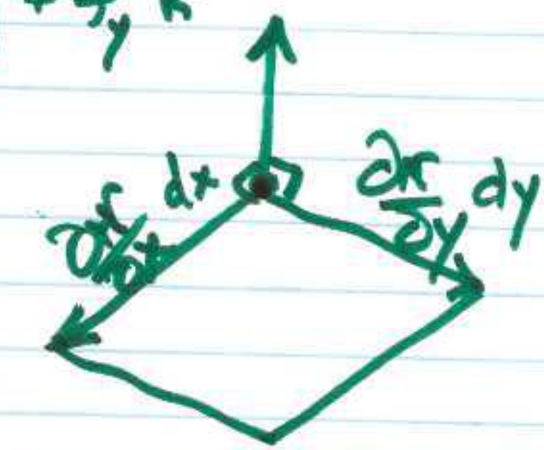
$$dS = \left| z_x \times z_y \right| dx\,dy$$

$$\mathbf{r} = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

generates surface S

$$\frac{\partial \mathbf{r}}{\partial x} = \underline{\hat{i} + f_x \hat{k}} \qquad z_x$$

$$\frac{\partial \mathbf{r}}{\partial y} = \hat{j} + f_y \hat{k}$$



$$dS = \left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| dy\,dx$$

area of differential "scale"

$$A_S = \iint_S dS = \iint_T \left| \underbrace{\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}}_{?} \right| dy\,dx$$

$$\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} =$$

$$\hat{i}(-f_x) - \hat{j}(f_y) + 1$$

Find norm $= \sqrt{f_x^2 + f_y^2 + 1}$

So final formula for A_s (surface integral)

$$A_s = \iint_T \underbrace{\sqrt{1 + f_x^2 + f_y^2}}_{\sec\theta} \, dxy$$

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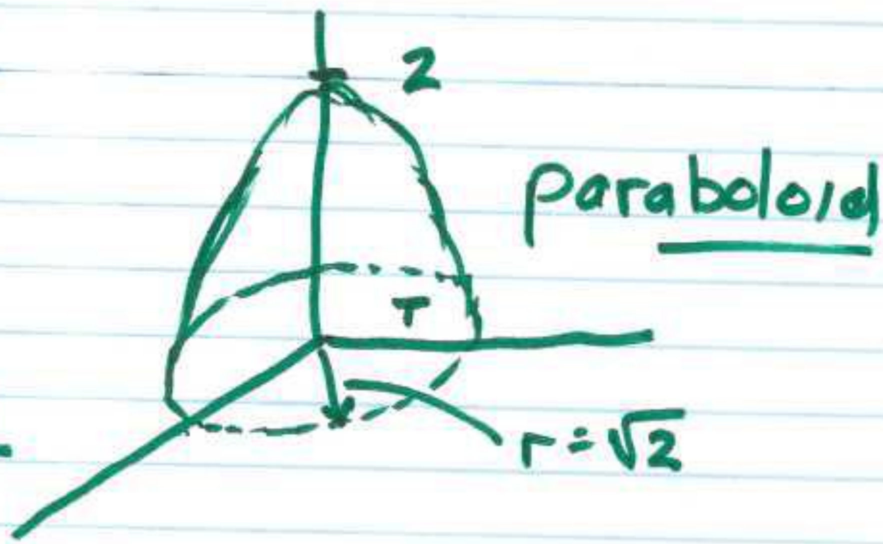
Prob: Given $z = 2 - x^2 - y^2$



$a = b\sec\theta$

$$-z = -f(x,y) = x^2+y^2-2$$

$$f(x,y) = 2-(x^2+y^2)$$

$$f_x = -2x \qquad f_y = -2y$$

$$\sqrt{1+4x^2+4y^2} = \sqrt{1+4(x^2+y^2)}$$

$$A = \iint_S \sqrt{1+4(x^2+y^2)}\,dx\,dy$$

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$x = r\cos\theta \quad y = r\sin\theta$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}\,\underline{r\,dr}\,d\theta$$

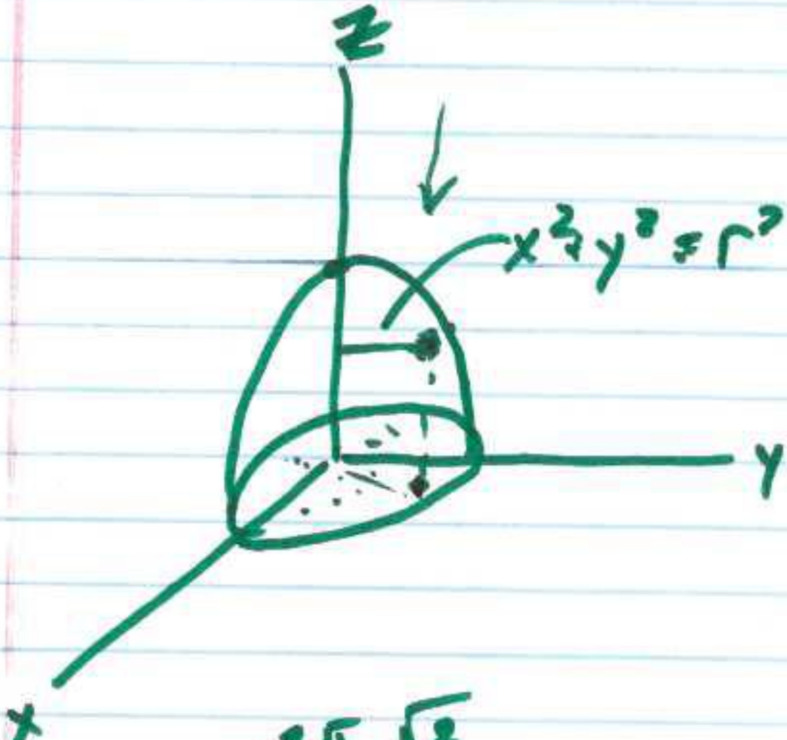
$u = 1+4r^2 \Rightarrow du = 8r\,dr$

r-int: $\frac{1}{8}\int_1^9 u^{1/2}\,du = \frac{1}{\cancel{8}\,4}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_1^9$

$= \frac{1}{12}(27-1) = \frac{26}{12}$

θ-int $\int_0^{2\pi}\frac{26}{12}\,d\theta = \cancel{2}\pi\cdot\frac{13}{\cancel{6}\,3} = \frac{13}{3}\pi$

6



$$z = 2 - x^2 - y^2$$

$$I_z = \int_0^{2\pi} \int_0^{\sqrt{2}} (x^2+y^2)\sqrt{1+4x^2+4y^2}\,dx\,dy$$

$$I_z = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} (r^2)\sqrt{1+4r^2}\,r\,dr\,d\theta$$

$u = 1+4r^2 \quad du = 8r\,dr$

⑦

r-int $\displaystyle\int_1^9 \left(\frac{u-1}{4}\right) u^{1/2} \frac{du}{8}$

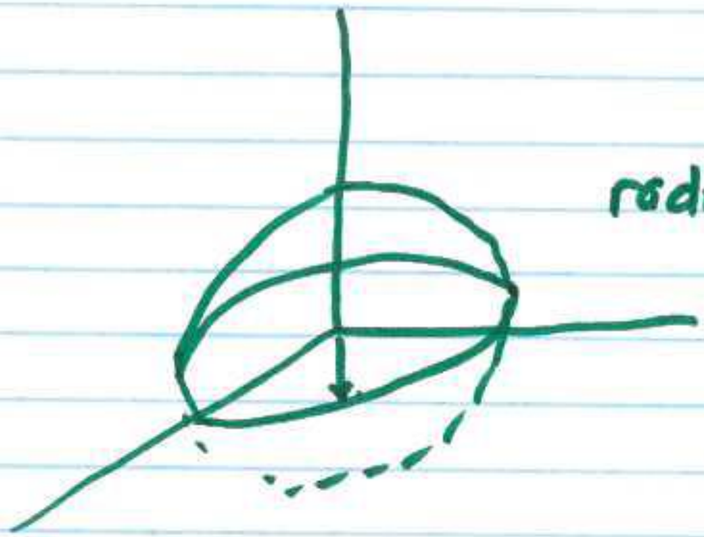
$\frac{6}{15} - \frac{10}{15} = -\frac{4}{15}$

$= \frac{1}{32} \displaystyle\int_1^9 \left(u^{3/2} - u^{1/2}\right) du = \frac{1}{32}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^9$

upper $\frac{\cancel{2}\pi}{\cancel{32}_{16}} \left[\frac{2}{5}(243) - 18\right] = \boxed{\frac{\pi}{16}\left(\frac{486}{5} - 18\right)}$

$I_z = \frac{1}{32}\left[\frac{486}{5} - 18 + \frac{4}{15}\right] 2\pi$

8



$r(x,y,z) \rightarrow$ spherical co-ords

$$\langle \overset{x}{a\sin\phi\cos\theta}, \overset{y}{a\sin\phi\sin\theta}, \overset{z}{a\cos\phi} \rangle$$

$0 \le \theta \le 2\pi \quad 0 \le \phi \le \pi$

$$\frac{\partial r}{\partial \phi} = \langle a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi \rangle$$

$$\frac{\partial r}{\partial \theta} = \langle -a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0 \rangle$$

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What is centroid $\bar{x}, \bar{y} = 0$

$$\iint_S \underbrace{(2-x^2-y^2)}_{z} \sqrt{1+4x^2+4y^2}\, dx\, dy$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{2}} (2-r^2)(1+4r^2)^{\frac{1}{2}} r\, dr\, d\theta$$

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{2}} 2\cdot(1+4r^2)^{\frac{1}{2}} r\, dr\, d\theta = \frac{26}{3}\pi$$

$$M_{xy} = \frac{26}{3}\pi - I_z$$

$$\bar{z} = \frac{M_{xy}}{A} = \frac{\frac{26}{3}\pi - I_z}{\frac{13}{3}\pi} = \bar{z}$$

9

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \hat{i}\left(a^2\sin^2\phi\cos\theta\right) - \hat{j}\left(-a^2\sin^2\phi\sin\theta\right) ?$$

$$+ \hat{k}\left(a^2\underline{\cos^2\theta}\sin\phi\cos\phi + a^2\underline{\sin^2\theta}\overset{\cos\phi}{\hat{}}\sin\phi\right)$$

$$|\cdot| = a^2\left[\underline{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta} + \sin^4\phi\cos^2\phi\right]$$

$$= a^2\left[\sin^4\phi + \overset{\sin^2\phi}{\cos^2\phi}\right]$$

$$\sin^2\phi\left(\sin^2\phi + \cos^2\phi\right) = \sin^2\phi$$

(16)

$$= a^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}$$

$$= a^2\sqrt{\sin^2\phi\,(\underbrace{\sin^2\phi + \cos^2\phi}_{1})}$$

$$a^2\sin\phi$$

$$A = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta =$$

$$a^2\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi} d\theta = a^2\int_0^{2\pi} 2\, d\theta =$$

$$2a^2\cdot 2\pi = 4\pi a^2$$