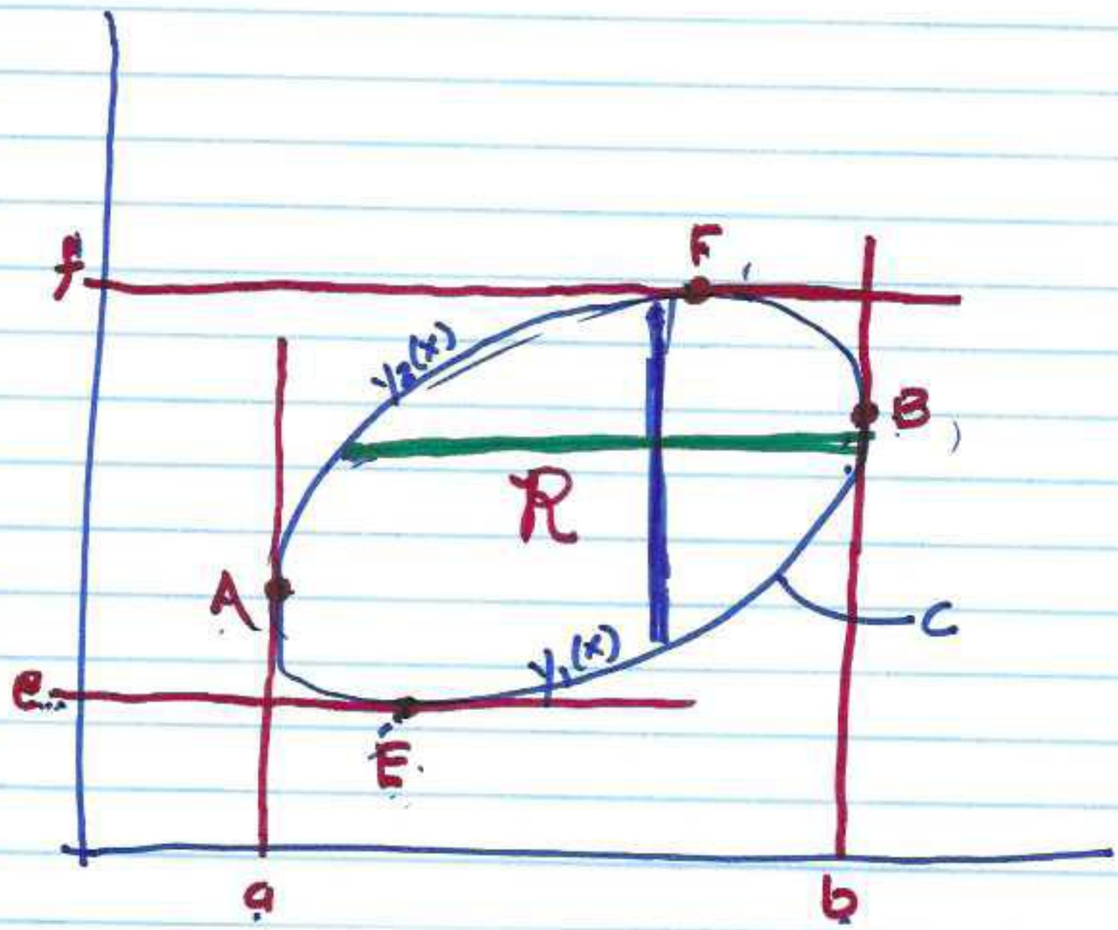
W9WZG

①

§16.4 Green's Thm (in the plane)

big picture:



$y = y_1(x)$ on AEB

$y = y_2(x)$ on AFB

$x = x_1(y)$ on EAF

$x = x_2(y)$ on EBF

$$\mathbb{F}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$

$$d\mathbf{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\oint \mathbb{F}\cdot d\mathbf{r} = \oint P\,dx + Q\,dy$$

Want to show

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \underline{\underline{\frac{\partial P}{\partial y}}}\right) dx\,dy$$

1^{st}

$$* \quad \iint_R \frac{\partial P}{\partial y} = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}\,dy\right] dx = \int_a^b \left[P(x,y_2) - P(x,y_1)\right] dx$$

$$- \int_a^b P(x,y_1)\,dx - \int_b^a P(x,y_2)\,dx = -\oint P\,dx$$

2nd part
$$\iint_R \underbrace{\frac{\partial Q}{\partial x}}\,dx\,dy = \int_e^f \left[\int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}\,dx\right] dy$$

$$= \int_e^f \left[Q(x_2, y) - Q(x_1(y)) \right] dy$$

change signs
$$\int_f^e Q(x_1, y)\,dy + \int_e^f Q(x_2, y)\,dy = \oint_C Q\,dy$$

$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy - \underline{\iint \frac{\partial P}{\partial y}\,dx\,dy} = \oint_C Q\,dy + \underline{P\,dx}$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxy = \oint_C P\,dx + Q\,dy \quad \blacksquare$$

④

Trick: Let $P = -y$; $Q = x$

$$\iint_R 1-(-1)\,dx\,dy = \oint -y\,dx + x\,dy$$

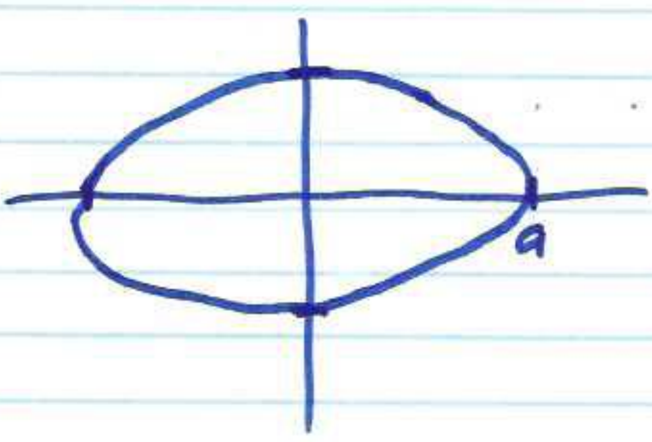
$$\downarrow$$

$$2A_R = \oint -y\,dx + x\,dy$$

So: $A_R = \frac{1}{2}\oint -y\,dx + x\,dy$

Ellipse : semi-major axis = a
semi-minor " = b

$x = a\cos\theta$; $y = b\sin\theta$ ✓



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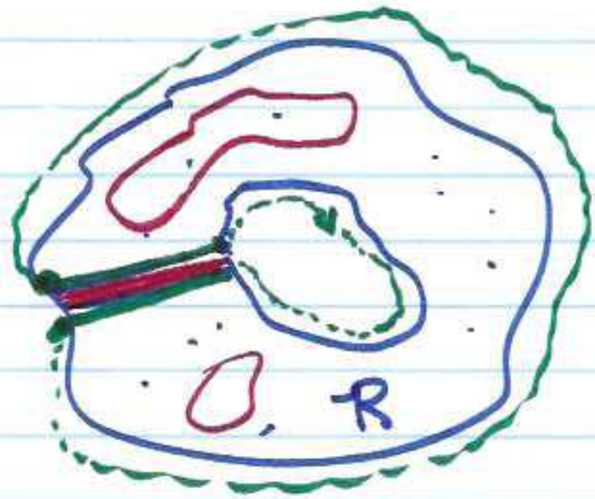
$$A_{\text{ellipse}} = \frac{1}{2}\oint -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_0^{2\pi} (-b\sin\theta)(-a\sin\theta) + (a\cos\theta)(b\cos\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (ab\sin^2\theta + ab\cos^2\theta)\,d\theta =$$

$$= \frac{1}{2}\int_0^{2\pi} (ab)\,1\,d\theta = \frac{2\pi ab}{2} = \pi ab$$

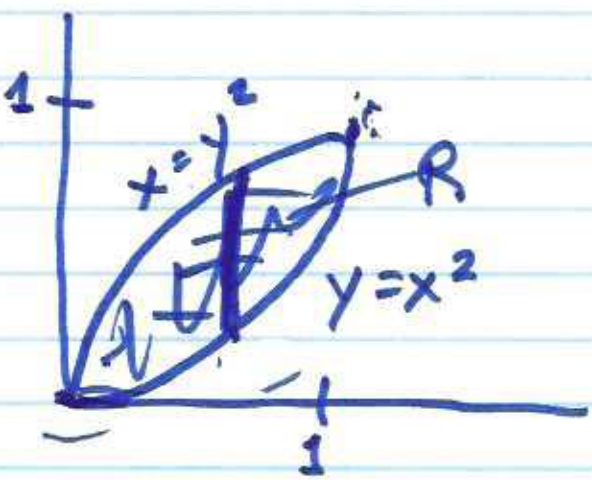
⑥



Prob: $\oint$ $P(x,y) = 2xy - x^2$

$Q(x,y) = x + y^2$

R:



$$\iint_R (1 - (2x))\, dx\, dy = \oint_{C_R} (2xy - x^2)\,dx + (x+y^2)\,dy$$

⑦

Lower $\oint_{C_R} (2xy - x^2)dx + (x+y^2)(2x\,dx)$

$= \int_0^1 (2xy - x^2 + 2x^2 + 2xy^2)\,dx$

$= \int_0^1 (2x^3 - x^2 + 2x^2 + 2x^5)\,dx = \int_0^1 2x^3 + x^2 + 2x^5\,dx$

$= \left[\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3}\right]_0^1 = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

$\underline{x = y^2}$

Upper $\oint = \int_1^0 (2y^3 - y^4)(2y\,dy) + (2y^2)(dy)$

$\int_1^0 \left[(2y^3 - y^4)(2y) + 2y^2\right] dy$

(8)

$$\int_1^0 (4y^4 - 2y^5 + 2y^2)\,dy$$

$$= \left[\frac{4}{5}y^5 - \frac{y^6}{3} + \frac{2}{3}y^3\right]_1^0$$

$$= -\left(\frac{4}{5} - \frac{1}{3} + \frac{2}{3}\right) = -\left(\frac{12}{15} - \frac{5}{15} + \frac{10}{15}\right) =$$

$$\underline{-\frac{17}{15}}$$

So total loop $\frac{7}{6} - \frac{17}{15} = \frac{35}{30} - \frac{34}{30} = \frac{1}{30}$

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$$\iint_R (1-2x)\,dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (1-2x)\,dy\,dx \Rightarrow$$

$$= \int_0^1 \Big[y - 2xy\Big]_{y=x^2}^{\sqrt{x}} dx = $$

$$\int_0^1 \Big[\left(\sqrt{x} - 2x^{3/2}\right) - \left(x^2 - 2x^3\right)\Big] dx$$

$$= \int_0^1 \left(x^{1/2} - 2x^{3/2} - x^2 + 2x^3\right) dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{4}{5}x^{5/2} - \frac{x^3}{3} + \frac{x^4}{2}\right]_0^1 \Rightarrow$$

$$\frac{2}{3} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2} = \frac{5}{6} - \frac{4}{5} = \frac{25}{30} - \frac{24}{30} = \frac{1}{30}$$