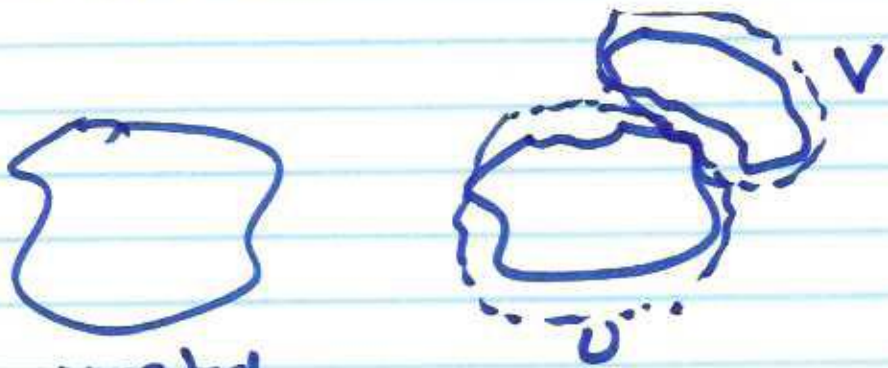
16.3§ Potential Theory:

We're dealing with open sets / domains

Connectedness :



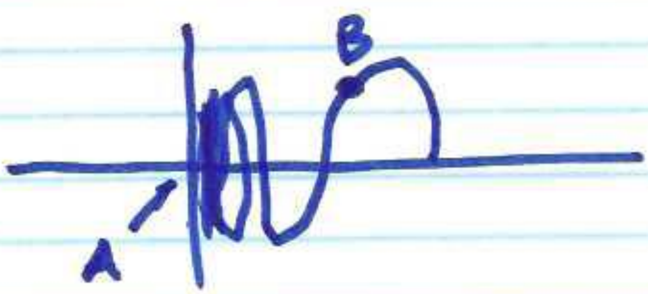
connected

$U \cap V = \emptyset$

Path-wise connectedness

pathwise connected →

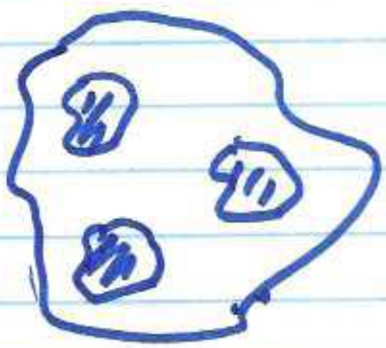
⇒ connected



$y = \sin\left(\frac{1}{x}\right) \quad x \neq 0$

$y = 1 \text{ if } x = 0$

Simply connected set "has no holes"



Use curl test for conservative fields

1. $yz\hat{i} + xz\hat{j} + xy\hat{k} = \mathbf{V}$

$$\underline{\nabla \times \mathbf{V}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$\hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z) = 0$$

4. $\mathbf{F} = -y\hat{i} + x\hat{j}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & +x & 0 \end{vmatrix} = ?$$

$$\hat{i}(0) - \hat{j}(0) + \hat{k}(1+1) = \underline{2\hat{k}}$$

Let $\mathbf{F} = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}$

then $\mathbf{F}$ is conservative if:

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \;\Big|\; \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \;\Big|\; \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

② $$\mathbf{F} = \overbrace{y\sin z}^{M}\,\hat{\imath} + \overbrace{x\sin z}^{N}\,\hat{\jmath} + \overbrace{xy\cos z}^{P}\,\hat{k}$$

(i) $x\cos z \overset{?}{=} x\cos z$ ✓

(ii) $y\cos z \overset{?}{=} y\cos z$ ✓

(iii) $\sin z \overset{?}{=} \sin z$ ✓

Huge Fact: Conservative fields are gradient fields

$\circled{5}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \underline{f}(B) - \underline{f}(A)$$

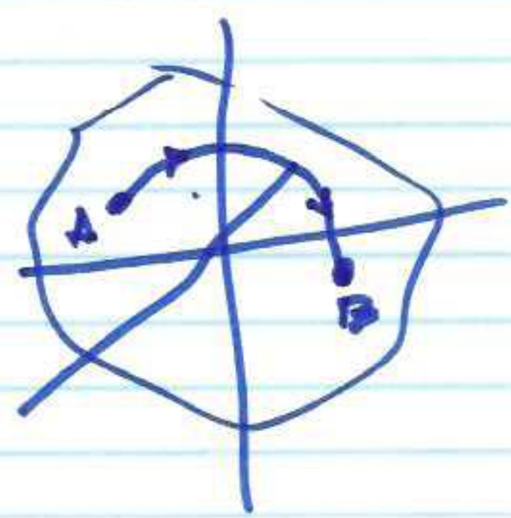
$\nabla f = \mathbf{F}$ ← gradient of potential

↑
potential function

Given $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

$$\int_A^B \mathbf{F} \cdot d\mathbf{r}$$

$$\Big[f(x,y,z)\Big]\Big|_A^B$$



⑥

~~A~~ ~~(0,2,2)~~ $(0,-2,1) = A$

$(1,3,-1) = B$

Then $\displaystyle\int_A^B \mathbb{F}\cdot d\mathbf{r} = \frac{1}{1+9+1} - \frac{1}{0^2+4+1} =$

$\underline{\frac{1}{11} - \frac{1}{5}} = \underline{?}$

Note: $\boxed{\displaystyle\int_C \mathbb{F}\cdot d\mathbf{r} = \int_A^B \nabla f\cdot d\mathbf{r}}$ (because $\mathbb{F} = \nabla f$)

$\mathbb{E} = \nabla\phi$

$\mathbb{F} = \nabla\phi$

$\oint_C \mathbb{F}\cdot d\mathbf{r} = 0$

$M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$

differential form

$\mathbb{F} = M\hat{i} + N\hat{j} + P\hat{k}$

$d\mathbb{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\underline{M\,dx + N\,dy + P\,dz}$$

$$\int_A^B \mathbb{F} \cdot d\mathbb{r} = \int_A^B M\,dx + N\,dy + P\,dz$$

$$\int_{x_A}^{x_B} M\,dx + \int_{y_A}^{y_B} N\,dy + \int_{z_A}^{z_B} P\,dz$$

If the differential form is <u>exact</u>

(refer to component test for gradient function)

If M, N, P pass component test for conservativeness, then $\exists f$

$df = M\,dx + N\,dy + P\,dz$ vastly simpler

②

Show that $\overset{M}{y}dx + \overset{N}{x}dy + \overset{P}{4}dz$ is an exact differential & integrate it over a path from $(1,1,1)$ to $(2,3,-1)$

Want $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$ | $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$ | $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

(i) $0 = 0$ ✓

(ii) $0 = 0$ ✓

(iii) $1 = 1$ ✓

So there exists some scalar function $f(x,y,z)$

such that $y\,dx + x\,dy + 4\,dz = df$

Integrate $\frac{\partial f}{\partial x} = y$ | $\frac{\partial f}{\partial y} = x$ | $\frac{\partial f}{\partial z} = 4$

$\int \frac{\partial f}{\partial x}\,dx = yx + g(y,z)$

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$$\frac{\partial f}{\partial y} = x + \underbrace{\frac{\partial g}{\partial y}}_{0} = x$$

implies $\Rightarrow$ $g(y, z) = h(z)$

$$f(x, y, z) = x + h(z)$$

$$\frac{\partial f}{\partial z} = 0 + \frac{\partial h}{\partial z} = 4 \Rightarrow h(z) = 4z + C$$

Finally $f(x, y, z) = \boxed{xy + 4z + C}$

potential function

So $f(x, y, z) = \underline{xy + 4z + C}$

$$\nabla f = y\hat{i} + x\hat{j} + 4\hat{k} \; (+\underbrace{C}_{0})$$