SV62G

①

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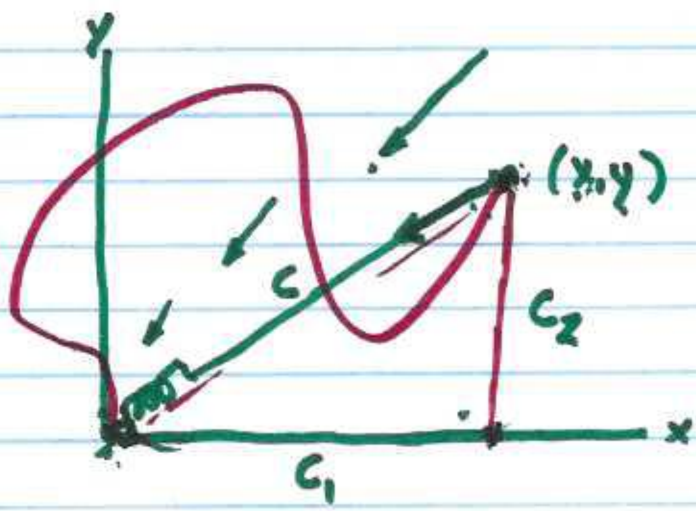
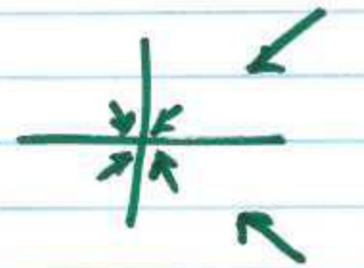
Integrals (line) in Vector Fields.

Conservative vector field.





$\mathbb{F} = -kx$

$\underline{\underline{y = mx}}$ $\quad \underset{\text{def'n}}{\int F\,ds} = \underset{\text{use for calc}}{\underline{\int F(t)\,|r'(t)|\,dt}}$

$C: \quad r(t) = t\hat{i} + mt\hat{j}$

$F(t) = -k(t\hat{i} + mt\hat{j})$

$* \ \mathbb{F}(x,y) = -k(x\hat{i} + y\hat{j})$

② $(x,y) = \mathbf{r}(1)$

$\mathbf{r}'(t) = \hat{\imath} + m\hat{\jmath}$

$\mathbb{F}(t)\cdot\mathbf{r}'(t) = -k\left(t + m^2 t\right)$

$$-k\int_0^1 (1+m^2)\,t\,dt = -k(1+m^2)\frac{t^2}{2}\Big|_0^1 =$$

$= \frac{-k}{2}(1+m^2)$ is work done by spring

C_1 C_2

Summary displacement along curve C results in doing $\frac{1}{2}k(1+m^2)$ J <u>against</u> spring

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Now try $C_1 + C_2$

$C_1 : \mathbf{r}_1(t) = t\hat{\imath} \qquad \mathbf{r}_1'(t) = \hat{\imath}$

Now form $\mathbb{F}(t)\cdot\mathbf{r}_1'(t) = -kt$

$$\int_0^1 -kt\,dt = -\frac{1}{2}kt^2\Big|_0^1 = \underline{-\frac{1}{2}k}$$

③

C_2 : $\mathbf{r}_2(t) = mt\hat{j}$ $\quad \mathbf{r}_2'(t) = m\hat{j}$

Form $\mathbf{F}(t)\cdot\mathbf{r}_2'(t) = (-kmt\cdot m) = -km^2t$

$$\int_0^1 -km^2\,\underline{t\,dt} = -\frac{k}{2}m^2t^2\Big|_0^1$$

$$-\frac{k}{2}m^2$$

Work required to traverse C_1, then C_2 is

$$\underbrace{-\frac{k}{2}}_{C_1} + \underbrace{\left(-\frac{k}{2}m^2\right)}_{C_2} = \underbrace{\underline{-\frac{k}{2}(1+m^2)}}_{= C}$$

④

Test for vector field to be conservative:

Given $\mathbb{F}(x,y,z)$, calculate the "curl" of $\mathbb{F}$.

$$\nabla \times \mathbb{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

If (and only if) $\nabla \times \mathbb{F} = 0$ (ie. curl $\mathbb{F}$) then $\mathbb{F}$ is conservative.

$$\nabla \times \mathbb{F}(x,y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -kx & -ky & 0 \end{vmatrix}$$

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$$\nabla \times \mathbb{F}(x,y) = \hat{i}\,0 + \hat{j}\,0 + \hat{k}\,(0-0) = 0$$

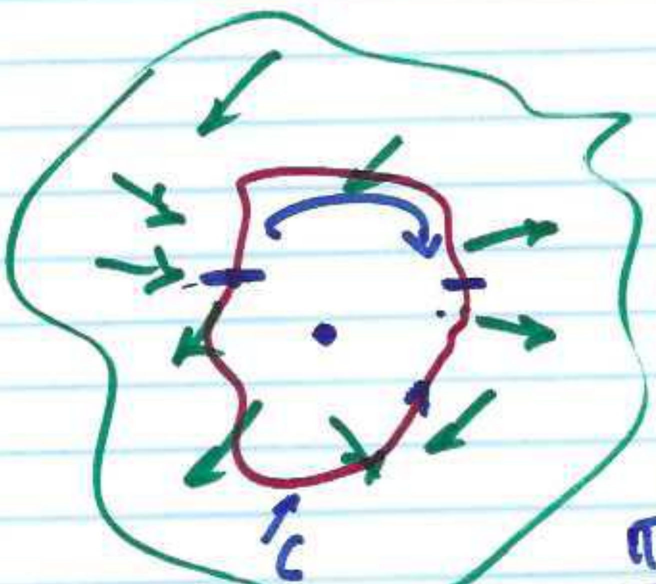
Borrow from 16.3

$$\mathbb{F}(x,y,z) = (e^x \cos y + yz)\,\hat{i} + (xz - e^x \sin y)\,\hat{j} + (xy+z)\,\hat{k}$$

$$\nabla \times \mathbb{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos y + yz & xz - e^x \sin y & xy+z \end{vmatrix}$$

(6)

$$\hat{i}\,\overbrace{(x-x)}^{0} - \hat{j}\,\overbrace{(y-y)}^{0} + \hat{k}\Big(\cancel{z} - e^x \sin y - \underbrace{\left[-e^x \sin y + \cancel{z}\right]}_{0}$$



$\mathbf{F}(x, y, z)$

$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{T}_c(t)$$

$\int \underline{\mathbf{F} \cdot \mathbf{T}\, ds}$ is called the

circulation integral of $\mathbf{F}$ along

closed path C

Ex: Velocity field is

$$\mathbf{F}(x,y,z) = x\hat{i} + z\hat{j} + y\hat{k}$$

Find flow along helix

$$\mathbf{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k} \quad (0 \le t \le \tfrac{\pi}{2})$$

Step 1: Parametrize helix - done

Step 2: " " field

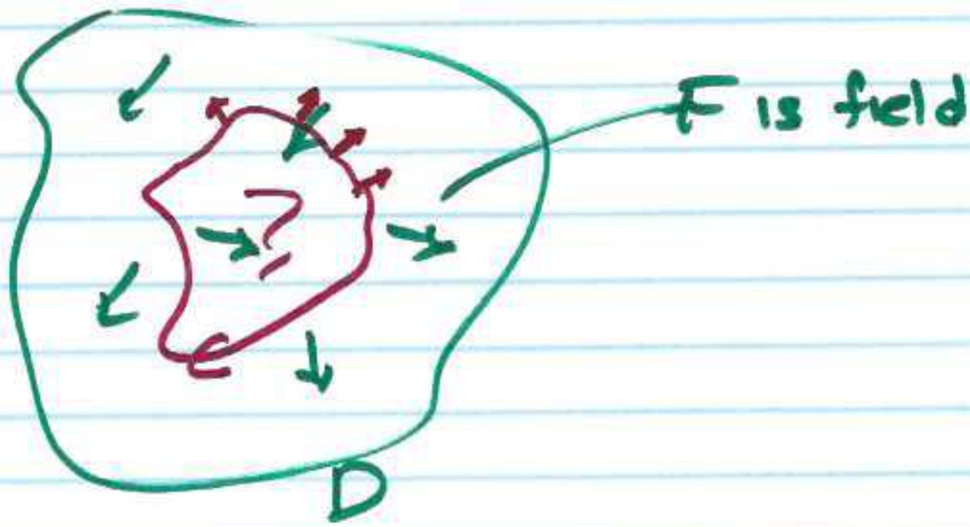
$$\mathbf{F}(t) = \cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{j}$$

$$\mathbf{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$

$$\int_0^{\pi/2} \left(-\sin t\cos t + t\cos t + \sin t\right) dt$$

$$= \left[\frac{\cos^2 t}{2} + t\sin t\right]_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left[\frac{1}{2} + 0\right] = \frac{\pi - 1}{2}$$

Flux integrals



Flux of $\mathbb{F}$ across C is:

$$\int_C \mathbf{F}\cdot \mathbf{n}\, ds$$

unit outward normal

If $\mathbb{F} = M\hat{i} + N\hat{j}$ $= \oint_C M\,dy - N\,dx$

Find flux of $\mathbb{F} = (x-y)\,\hat{i} + x\,\hat{j}$ across

$x^2+y^2=1$

$\mathbb{r}(t) = \cos t\,\hat{i} + \underline{\sin t}\,\hat{j}$

$M : \; x-y = \cos t - \sin t$

$N : \; x = \cos t$

$$\oint_0^{2\pi} \cos t - \sin t \;\; \cos t\,dt \; -$$

$$\oint_0^{2\pi} \cos t\,(-\sin t)\,dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt - \underline{\int_0^{2\pi} \sin t\cos t\,dt + \int_0^{2\pi} \sin t\cos t\,dt}$$

$= \pi$