

G8RDB

4/1

Review:

$$\iiint f(x, y, z) dx dy dz$$

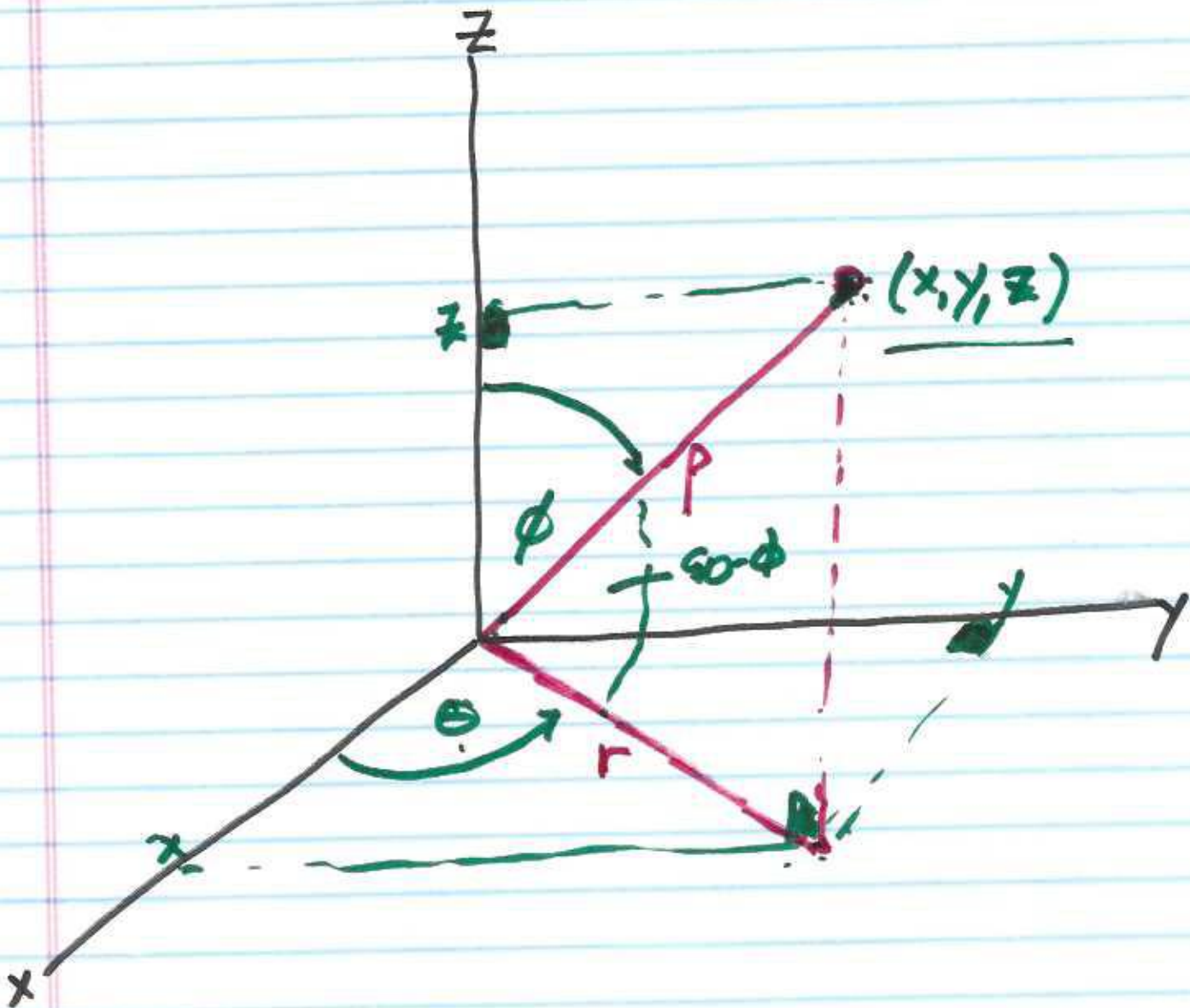
$$\iiint f(r, \theta, z) r dr d\theta dz$$

$$\iiint f(\rho, \theta, \phi) \rho^2 \sin \phi d\rho d\theta d\phi$$

(Note: In the original image, an arrow points from r in the middle equation to ρ in the bottom equation, and another arrow points from $\rho^2 \sin \phi$ in the bottom equation to the volume element $d\rho d\theta d\phi$.)

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \rho^2 \sin \phi$$

(2)



ρ = radius

θ = azimuth

ϕ = declination

③

$$(x, y, z) \mapsto (\rho, \theta, \phi)$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

because: $r = \sqrt{x^2 + y^2}$ and

$$\rho = \sqrt{r^2 + z^2}.$$

$$\phi = \rho \cos(90 - \phi) = r$$

$$r = \rho \sin \phi$$

$$\frac{r}{\rho} = \sin \phi, \text{ so } \phi = \arcsin\left(\frac{r}{\rho}\right)$$

$$\arcsin \sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}}$$

$$\theta = \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

$$x = r \cos \theta$$

$$\text{so } \theta = \arccos\left(\frac{x}{r}\right)$$

$$(\rho, \theta, \phi) \mapsto (x, y, z) \quad \textcircled{1}$$

$$x = \overbrace{\rho \sin \phi}^r \cos \theta$$

$$y = \rho \sin \phi \sin \theta \quad \leftarrow$$

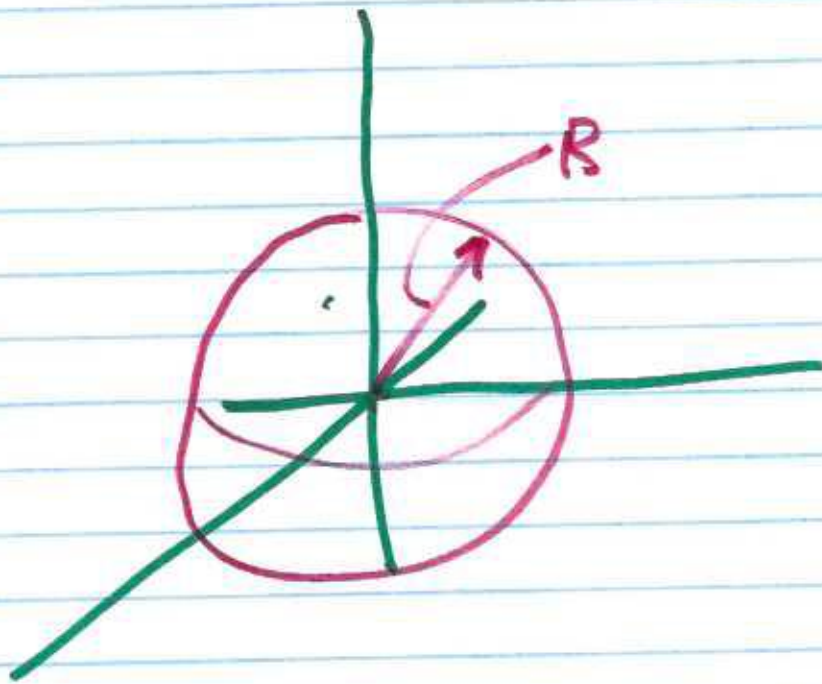
$$z = \rho \cos \phi$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$\frac{\partial y}{\partial \theta} = \rho \sin \phi \cos \theta$$

all reduces to $\rho^2 \sin \phi$

(5)



Vd of Sphere

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3} \right]_0^R \sin\phi \, d\phi \, d\theta$$

⑥

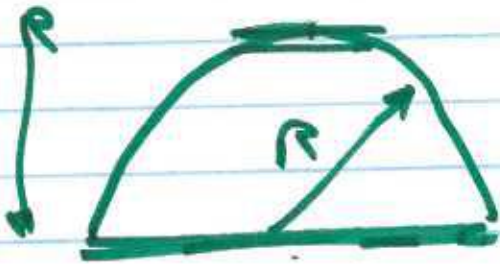
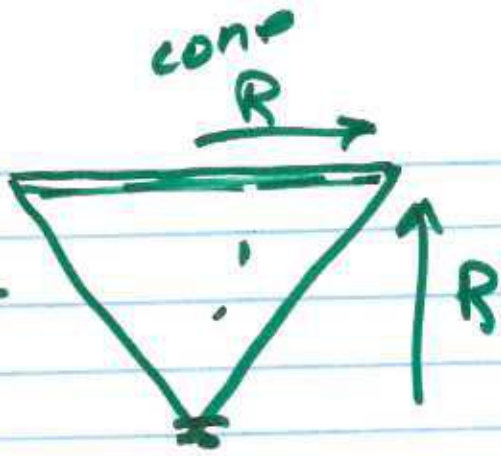
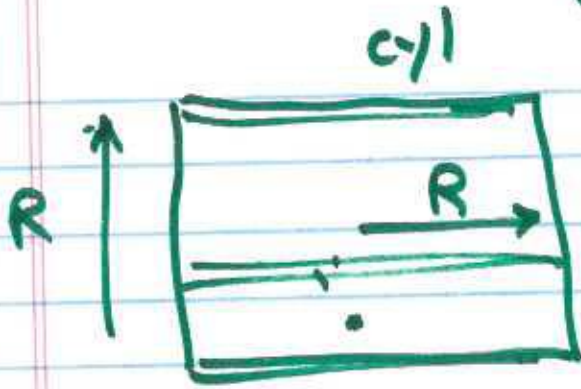
$$\int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta \quad \curvearrowright$$

$$\int_0^{2\pi} [-\cos \phi]_0^{\pi} \, d\theta \quad \curvearrowright$$

$$\int_0^{2\pi} (1+1) \, d\theta = R^3 \int_0^{2\pi} 2 \, d\theta \quad \curvearrowright$$

$$= R^3 \cdot 2 \cdot 2\pi = \boxed{\frac{4}{3} \pi R^3}$$

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8

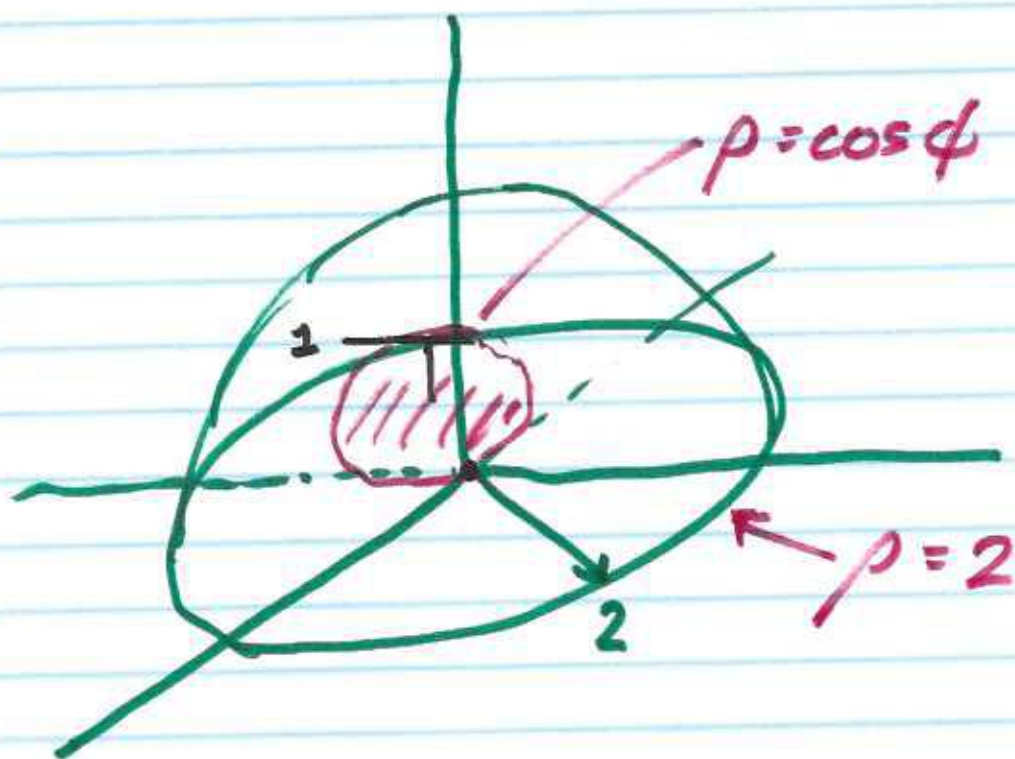
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$$\rho = \cos \phi$$

sphere

$$\rho = 2 \quad z \geq 0$$

hemisphere



$$\frac{2}{3} \pi (8) = \frac{16}{3} \pi \text{ vol hemi}$$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \leftarrow \text{sphere}$$

(9)

Vol interior sphere $\int_0^{2\pi} \int_0^{\pi} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$\int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3} \right]_0^{\cos\phi} \sin\phi \, d\phi \, d\theta$$

$$\frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \cos^3\phi \sin\phi \, d\phi \, d\theta$$

$$\int \cos^3\phi \sin\phi \, d\phi \quad \text{let } \cos\phi = u$$

$$\int u^3 (-du) = -\frac{u^4}{4}$$

(10)

$$\text{Vol} = \frac{1}{3} \int_0^{2\pi} \left[\underbrace{\left(-\frac{1}{4}\right) \cos^4 \phi}_0 \right]_0^{\pi} d\theta$$

$$= \frac{1}{6} \int_0^{2\pi} d\theta = \left(\frac{\pi}{3}\right)$$

$$\frac{16}{3}\pi - \frac{\pi}{3} = \left(5\pi\right) \text{ final answer}$$

entire prob.
