MY6AU

late 11/10 (1)

→ A sequence is a mapping from integers to a set $f: \mathbb{N} \to X$

So... $\{x_0, x_1, x_2, \cdots x_n \cdots\} = \{x_n\} = \{x_n\}_{n=1}^{\infty}$

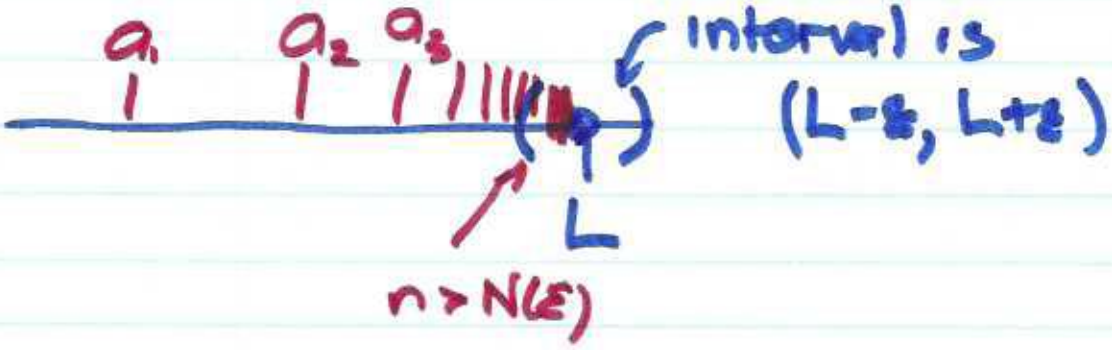
Real sequence $\{a_n\}$ where $a_n \in \mathbb{R}$.

Ex. $\mathbb{Z} = \{n\}_{n=1}^{\infty}$ $(a_{n+1} = a_n + 1)$ rule.

Functional sequence $\{x^n\}_{n=1}^{\infty}$ ...

Convergence / Divergence

We say $\{a_n\}$ approaches limit L as $n \to \infty$ if given $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that whenever $n > N(\varepsilon)$ we have $|a_n - L| < \varepsilon$.



If $\alpha < \varepsilon$ and $\varepsilon > 0$ for all ε, then

$\alpha = 0$.

The sequence diverges if it does not converge.

Two cases:

(i) difference between terms stays finite.

$1, 2, 3, 4, \cdots n, \cdots$

(ii) oscillation

$-1, 1, -1, 1, -1, 1, \cdots (-1)^n$

Rules for Manipulating Sequences:

Suppose $\{a_n\}$ converges to A and $\{b_n\}$ converges to B. Then:

(i) $\lim_{n\to\infty} \{a_n + b_n\} = A + B$

(ii) $\lim_{n\to\infty} \{a_n - b_n\} = A - B$

(iii) $\lim_{n\to\infty} \{k a_n\} = kA$ (k constant)

(iv) $\lim_{n\to\infty} \{a_n \cdot b_n\} = A \cdot B$

(v) $\lim_{n\to\infty} \left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}$ $B \neq 0$

(vi) $\lim_{n\to\infty} \{a_n^k\} = A^k$

Pitfall: $\{a_n\} \neq \{b_n\}$ can both diverge but $\{a_n \pm b_n\}$ converges

$a_n = (-1)^n \quad b^n = (-1)^{n+1}$

a_n: $-1, 1, -1, 1, -1, 1 \cdots$

$+$

b_n: $1, -1, 1, -1, 1, -1 \cdots$

$0, 0, 0, 0, 0, 0 \cdots$ converges

note: convergent + divergent = divergent

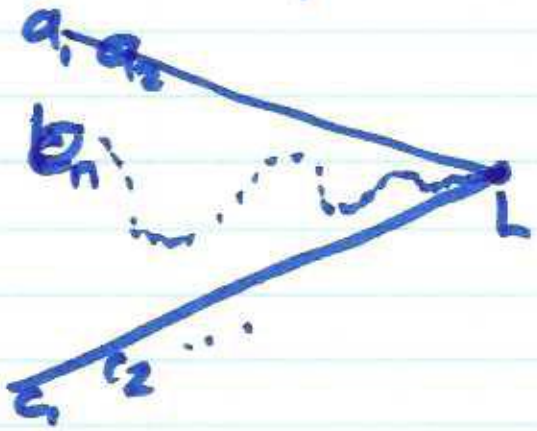
Sandwich Th^m:

Given $\{a_n\}, \{b_n\}, \{c_n\}$ with

$\{a_n\}$ & $\{c_n\}$ converging both to L,

if $a_n \geq b_n \geq c_n$ for all n, then

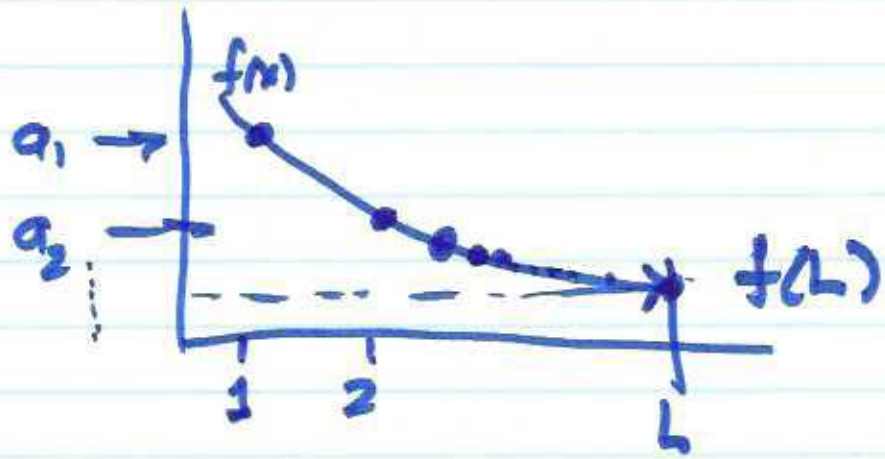
$\{b_n\}$ converges to L



Continuous Function Th^{m} (for sequences)

If $f(x)$ is continuous at $x = L$

and $f(n) = x_n$, then $\lim_{n\to\infty} = f(L)$



L'Hôpital's Rule for Sequences

Convention is to "continualize" the discrete indices.

Ex: Find $\lim_{n\to\infty} \frac{\ln n}{n} \to \frac{\infty}{\infty}$

By L'Hôpital $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \left(\frac{1/x}{1}\right) = 0$

(6)

Let $a_n = \left(\frac{n+1}{n-1}\right)^n$ What is $\lim_{n\to\infty} a_n$?

$\ln a_n = n \cdot \ln\left(\frac{n+1}{n-1}\right) \quad (\infty \cdot 0)$

$= \dfrac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \to \dfrac{0}{0}$

$\dfrac{\frac{1}{\left(\frac{n+1}{n-1}\right)} \cdot \frac{(n-1)(1) - (n+1)(1)}{(n-1)^2}}{-1/n^2}$

$(-n^2)\left[\frac{(n-1)}{(n+1)} \cdot \frac{-2}{(n-1)^2}\right] = \frac{+2n^2}{n^2-1} = \frac{+2}{1-\frac{1}{n^2}}$

$= 2$ so. $\lim_{n\to\infty} \ln a_n = 2 \Rightarrow$

$\lim_{n\to\infty} a_n = e^2$

Some exceptional sequences:

1. $a_n = \frac{\ln n}{n}$ $\quad L = 0$
2. $a_n = \sqrt[n]{n}$ $\quad L = 1$
3. $a_n = x^{1/n}$ $\quad L = 1$
4. $a_n = x^n$ $\quad L = 0 \quad |x| < 1$
5. $a_n = \left(1 + \frac{x}{n}\right)^n$ $\quad L = e^x$
6. $a_n = \frac{x^n}{n!}$ $\quad L = 0$

A sequence is non-decreasing if $a_{n+1} \geq a_n$

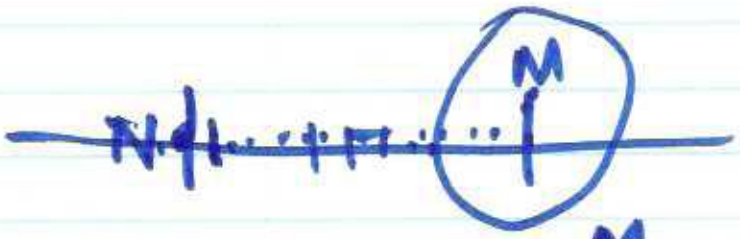
non-increasing if $a_{n+1} \leq a_n$

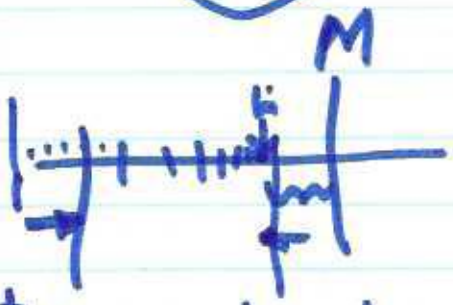
If a sequence is either non-increasing or non-decreasing it is called <u>monotonic</u>

Boundedness.

$\{a_n\}$ is bounded above if $a_n \leq M$

" " " below if $a_n \geq N$

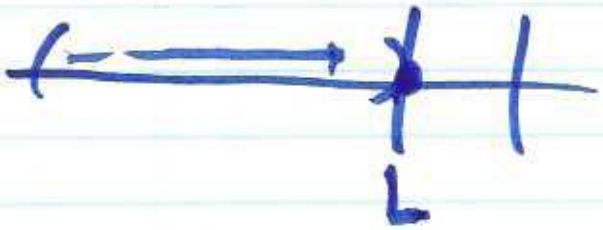




greatest lower bound

least upper bound

Th^{m}/ Monotone Convergence Th^{m}



$a_1, a_2, a_3, a_4 \cdots a_n$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 \cdots a_n$

$$\sum_{n=1}^{\infty} a_n := \lim_{n \to \infty} S_n$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{n+1}} + \cdots$$

$$0$$

$$S - \frac{1}{2}S = 1$$

$$\frac{1}{2}S = 1 \Rightarrow \boxed{S = 2}$$

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots}_{\geq \frac{1}{2}}$$

$\geq 1 \quad \geq \frac{1}{2}$

$\left\{\frac{1}{n}\right\}$

Problem: $\int \frac{dx}{(1+x^2)^2} = ?$

known: $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

Leibnitz Rule: $\int \frac{\partial}{\partial a}\left(\frac{1}{a^2+x^2}\right)dx = \int -\frac{2a}{(a^2+x^2)^2}\,dx$

$\frac{d}{da}$: $\frac{d}{da}\left[\frac{1}{a}\arctan\frac{x}{a}\right] =$ ↙

$$-\frac{1}{a^2}\arctan\frac{x}{a} + \frac{1}{a}\left[-\frac{x}{a^2+x^2}\right]$$

Set $a=1$: $-2\int \frac{dx}{(1+x^2)^2} = -\arctan x - \frac{x}{1+x^2}$

Conclude: $\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left[\arctan x + \frac{x}{1+x^2}\right]$

(2)

Check: $\frac{1}{2}\left[\arctan x + \frac{x}{1+x^2}\right]' = ?$

$$\frac{1}{2}\left[\left(\frac{1}{1+x^2}\right) + \left(\frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}\right)\right] = ?$$

$$\frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2} = ?$$

$$\frac{1+x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2} \checkmark$$