SCKRS

early 11/10 C

Sequences - ordered list of terms, either numerical or functions...

could be finite or infinite

Ex: finite $\{1, 2, 3\}$

infinite $\{1, 2, 3, \cdots n, \cdots\}$

Typical to represent an arbitrary sequence

$\{a_0, a_1, a_2, \cdots a_n, \cdots\}$

- $\{a_n\}$ $\quad a_n = 2^n$
- $\{1, 2, 4, 8, \cdots 2^n\}$

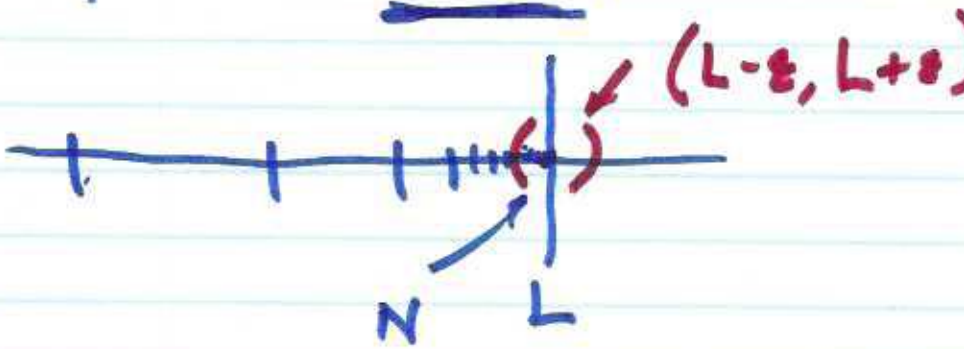
$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$

Two issues with sequences:

A sequence may converge or diverge

Convergence: We say $\{a_n\}$ converges to the limit L if for any given tolerance $\varepsilon > 0$

$|a_n - L| < \varepsilon$ for $n > N$.



For example: I claim $\left\{\frac{1}{2^n}\right\}$ converges to zero. Given $\varepsilon > 0$, say $\varepsilon = .001$. Can we find an N such that whenever $n > N$, $\left|\frac{1}{2^n} - 0\right| < .001$

Claim: Let $N = 10$, then $\frac{1}{2^{10}} = \frac{1}{1024} < \frac{1}{1000}$, so $n > N$ means $\left|\frac{1}{2^n} - 0\right| < .001$

Divergence - a sequence diverges if it does not convergence.

NB: Divergence can happen two ways:

1) Sequence has arbitrarily large gaps for all n. Consider $\{n\}_{n=1}^{\infty}$

2) Sequence can oscillate

Consider $\{(-1)^n\}_{n=1}^{\infty}$ $-1, 1, -1, 1, \ldots$

Rules for Sequences, Given $\{a_n\} \& \{b_n\}$

① Sum if $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$

then $\lim_{n\to\infty} \{a_n + b_n\} = A + B$

② Difference, as above

$\lim_{n\to\infty} \{a_n - b_n\} = A - B$

⑤ Constant Multiple, as above

$$\lim_{n\to\infty}\{ka_n\} = kA \quad , k \text{ constant}$$

④ Product Rule, as above

$$\lim_{n\to\infty}\{a_n \cdot b_n\} = AB$$

⑤ Quotient, as above with $B \neq 0$

$$\lim_{n\to\infty}\left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}.$$

Ex: $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = ?$

$\downarrow$

$\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1-0 = \frac{1}{2}$ = Limit

$a_n - b_n$

(5)

What is $\lim_{n\to\infty}\left(\frac{4-7n^6}{n^6+3}\right) = ?$

$$\left(\frac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}}\right) \to \left(\frac{-7}{1}\right) = -7$$

Some tricky points:

If $\{a_n\}$ & $\{b_n\}$ converge, so does the sum

BUT If $\{a_n\}$ & $\{b_n\}$ diverge, sum may converge

a_n: 1, -1, 1, -1, 1, -1 ...

+

b_n: -1, 1, -1, 1, -1, 1 ...

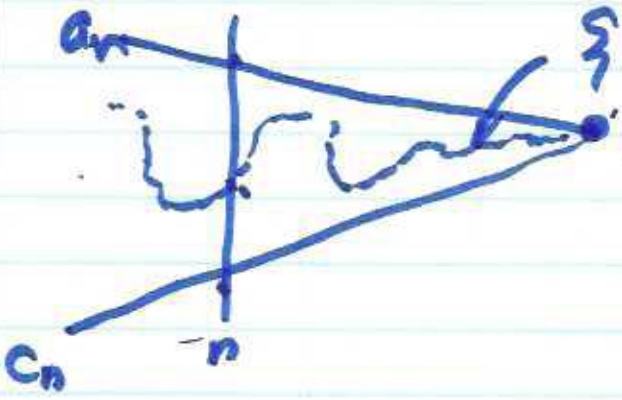
0, 0, 0, 0, 0, 0, ... → Limit = 0

Sandwich Thm

If $\{a_n\}, \{b_n\}, \{c_n\}$ and $a_n \le b_n \le c_n$

What can you say about $\{b_n\}$ if both

$\{a_n\}$ & $\{c_n\}$ converge to the same limit?



Continuous Function Th^m (for sequences)

Let $\{a_n\}$ converge to L.

If $f(x)$ is continuous @ L. Then

$$f(a_n\} \to f(L)$$

Ex: $\left\{\frac{1}{2^n}\right\} \to 0$ as $n \to \infty$

$f(x) = x^3$. This function is cont. @ 0

So... $\left\{\frac{1}{2^{3n}}\right\}$ converges to 0.

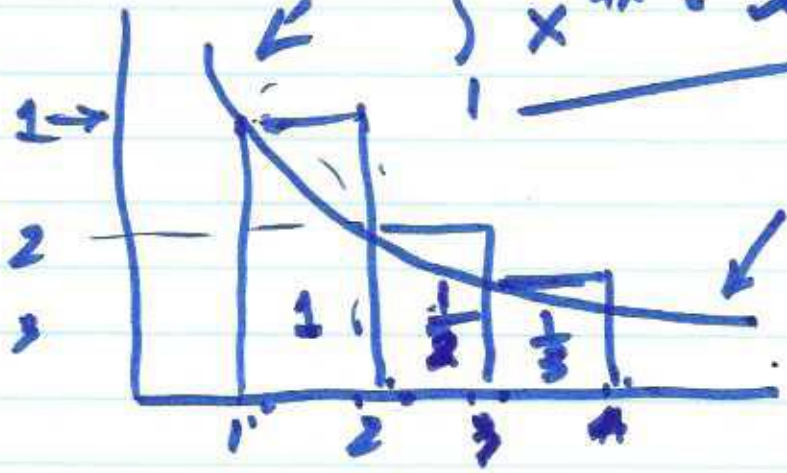
(7)

Thm Let $f(x)$ be defined for all $x > n_0$

Let $\{a_n\}$ be sequence such that $a_n = f(n)$

Then $\lim_{n\to\infty} a_n = L$ iff $\lim_{x\to\infty} f(x) = L$

$$\int_1^\infty \frac{1}{x}\,dx = \ln x \Big|_1^\infty = \infty$$



$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ diverges to ∞

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} \rightsquigarrow \frac{0}{0}$ or $\frac{\infty}{\infty}$

Try $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} \rightsquigarrow L$

$\frac{\ln x}{x}$

Use this on $\frac{\ln n}{n} \rightsquigarrow ?$

Try L'Hôpital on $\frac{\ln x}{x}$

$\lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$ → so differentiate:

$$\lim_{x\to\infty} \left(\frac{1/x}{1}\right) = \lim_{x\to\infty} \left(\frac{1}{x}\right) = 0$$

So we would confirm $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

(9)

$$a_n = \left(\frac{n+1}{n-1}\right)^n \qquad (1^\infty)$$

Does $\{a_n\}_{n=1}^{\infty}$ converge?

$$\ln a_n = n \ln\left(\frac{n+1}{n-1}\right)$$

$$\infty \cdot 0 \qquad \frac{0}{0}$$

$$= \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad \begin{matrix} \to 0 \\ \to 0 \end{matrix} \qquad \frac{2n^2}{n^2-1}$$

now a proper L'Hopital indeterminate

$$\lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} = \lim_{n\to\infty} \frac{2n^2/n^2-1}{1} = 2$$

$$\lim_{n\to\infty} a_n = e^2$$

Bounded Sequences :

1) A sequence $\{a_n\}$ is bounded from above if there exists a number M such that $M \geq a_n$, for all n.

2) A sequence $\{a_n\}$ is bounded from below if there exists a number N such that $N \leq a_n$ for all n.

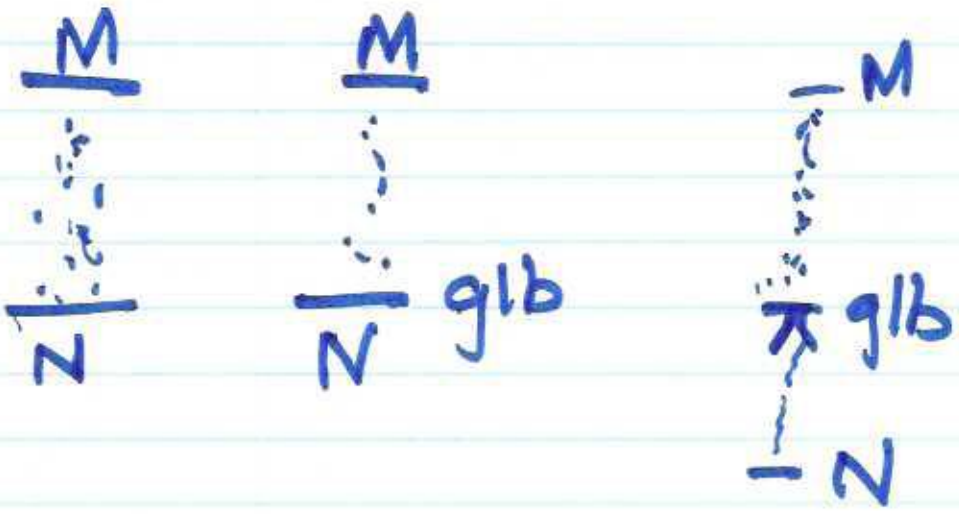
3) A sequence is <u>bounded</u> if

$$N \leq a_n \leq M.$$

4) A greatest lower bound is :

(i) a lower bound

(ii) if there is another lower bound, it is greater than or equal to it



5) least upper bound



A sequence is <u>non-decreasing</u> if

$a_n \leq a_{n+1}$

and it is non-increasing if

$a_n \geq a_{n+1}$

The sequence is <u>monotone</u> if either case

Monotone Convergence Theorem:

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.



$$\sum_{n=1}^{\infty} a_n$$

S_1 a_1

S_2 $a_1 + a_2$

S_3 $a_1 + a_2 + a_3$

$\downarrow$

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