

Cosets

Given group G , and a subset S of elements of G

we say gS (and Sg) are cosets of S in G .

$g \in G$.



Important case is when $H \leq G$ then

gH & Hg are cosets of H in G .

Number of elements in gH is indicated by $|gH|$

$eH = H$. Only coset that is a subgroup

Notation: $gH := \{gh : h \in H\}$

$Hg := \{hg : h \in H\}$

(2)

Examples of cosets.

1) Let $G = S_3 (\cong D_6)$

Pick $H \leq S_3$ to be $\{(1), (13)\}$

What are possible cosets?

NB: we call the g in gH or Hg the "coset representative".

$$S_3 = \{(1), (12), (13), (23), (123), (132)\}$$

1) $(1)H = H$ ✓ $H = \{(1), (13)\}$

2) $\underline{(12)}H = \{(12), (12)(13)\} = \{(12), \underline{(132)}\}$

$\underline{\{(132)H\}}$

3) $(13)H = \{(13)(1), (13)(13)\} = H$
 $\quad \quad \quad \downarrow \quad \quad \downarrow$
 $\quad \quad \quad \underline{(13)}, (1)$

③

$$4) (23)H = \{(23)(1), (23)(13)\} = 2$$

$$\{(23), (123)\} = (123)H$$



$$G = D_4 \text{ (order 8)}$$

$$\text{Let } H = \{R_0, R_{180}\}$$

$$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, V, H, D, D'\}$$

$$H \leq D_4$$

$$1) R_0 H = H$$

$$2) R_{90} H = \{R_{90}, R_{270}\} = R_{270} H \downarrow$$

$$\{R_{270}, R_{90}\}$$

$$3) R_{180} H = \{R_{180}, R_0\} = H$$

$$4) V H = \{V, H\}$$

$\uparrow$ subgp $\uparrow$ horiz flip

④

$$5) DH = \{D, D'\}$$

$$G = \mathbb{Z}_9 \quad H \leq G, H = \{0, 3, 6\}$$

$$1) 0 + H = \{0, 3, 6\} = 3 + H = 6 + H$$

$$2) 1 + H = \{1, 4, 7\} = 4 + H = 7 + H$$

$$3) 2 + H = \{2, 5, 8\} = 5 + H = 8 + H$$

$$3 \mid |\mathbb{Z}_9|$$

$$|H| \cdot [G:H] \quad \swarrow \text{index of } H \text{ in } G$$

$$3 \cdot 3 = 9 = |\mathbb{Z}|$$

$$|G| = |H| [G:H]$$

$$9 = 3 \cdot 3$$

(5)

Facts about cosets: Let $H \leq G$, $a, b \in G$

1) $a \in aH$ note $e \in H$, so $ae = a \in H$

2) $aH = H$ iff $a \in H$

$\Rightarrow$ if $a \in H$, then $aH = H$ —

$\Leftarrow$ if $aH = H$, then $a \in H$ —

3) $abH = a(bH)$ and $Hab = (Ha)b$ —

$$\{bh_1, bh_2, \dots, bh_n\} = bH$$

$$\{abh_1, abh_2, \dots, abh_n\} = a(bH)$$

$$\{abh_1, abh_2, \dots, abh_n\} = abH$$

4) $aH = bH$ iff $a \in bH$

Suppose $aH = bH \Rightarrow ah_1 = bh_2$

$$ah_1h_1^{-1} = bh_2h_1^{-1}$$

$$a = b(\underline{h_2h_1^{-1}}) \in bH$$

⑥

Suppose $a \in bH$

$$ah \in bHh \Rightarrow ah \in bH$$
$$\underline{aH = bH}$$

Note: $HH = H$

$$1H = \{h_1 h_2 : h_1, h_2 \in H\} = H$$

5) Either $aH = bH$ or $aH \cap bH = \emptyset$

Suppose $x \in aH \cap bH$ so

$$x \in aH \text{ and } x \in bH$$

$$\downarrow \qquad \qquad \downarrow$$
$$xH = aH \qquad xH = bH$$



same

* cosets are either identical or disjoint

$$6) \underline{aH = bH} \text{ if } a^{-1}b \in H$$

$$\underline{a^{-1}aH} = a^{-1}bH$$

$$H = (a^{-1}b)H$$

$$a^{-1}bH = H$$

$$\underline{aa^{-1}bH} = aH$$

⑦

$$7) |aH| = |bH|$$

$$\underline{ah \xrightarrow{\phi} bh}$$

$$\phi: aH \rightarrow bH$$

$$ah_1 = bh_2$$

$$\underline{ah_1 = bh_1}$$

$$ah_1 = bh_2$$

$$8) \underline{aH = Ha} \text{ iff } \underline{H = aHa^{-1}}$$

$$\underline{Ha} = \underline{aHa^{-1}} \cdot \underline{a} = \underline{aH}$$

$$9) aH \leq G \text{ iff } a \in H$$

Cosets partition a group

$$H \leq G \text{ then } G = a_1H \cup a_2H \cup \dots \cup a_nH$$

Partition:

Set of "blocks" B_i such that

$$B_i \neq \emptyset \forall i, B_i \cap B_j = \emptyset \text{ if } i \neq j,$$

$$\bigcup_i B_i = G$$